Plenum 1/11-13

8.2: 1, 5

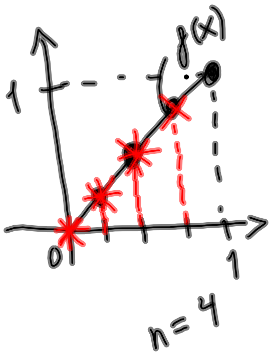
8.3: 1^{h}, 3abcdeg, 4^{d}, 5, 6, 7ab, (9)

8.4: 1^{e}, 2^{c}, 3^{b}, 5

8.5: 1a, 2, 4, 5

8.2: 5.) $f : [0,1] \to \mathbb{R}$, $f(x) = x$,

$$\Pi_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, 1\right\}, \quad \Delta x = \frac{1}{n}$$



a) $$\Phi(\Pi_n) = \underbrace{\frac{1}{n}}_{\Delta x} \sum_{k=1}^{n} \frac{k}{n}$$

$$= \frac{1}{n^2} \sum_{k=1}^{n} k \underset{\text{Opg 1.2.1}}{=} \frac{1}{n^2} \frac{n^2+n}{2} = \frac{1}{2}\left(1 + \frac{1}{n}\right)$$

$$N(\Pi_n) = \frac{1}{n} \sum_{k=0}^{n-1} \frac{k}{n} = \frac{1}{n^2} \sum_{k=0}^{n-1} k$$

$$= \frac{1}{n^2} \sum_{k=1}^{n-1} k \underset{\text{Oppg 1.2.1}}{=} \underline{\underline{\frac{1}{n^2}\left(1 - \frac{1}{n}\right)}}$$

litt regning

b) <u>Øvre integral:</u> $\overline{\int_0^1} x\,dx = \inf\{Ø(\Pi) : \Pi \text{ partisjon av } [0,1]\}$

$$\underset{\text{Korollar 8.2.4}}{=} \inf\{Ø(\Pi_n) : n \in \mathbb{N}\}$$

$$= \inf_{n \in \mathbb{N}} \left\{\frac{1}{2}\left(1 + \frac{1}{n}\right)\right\} = \underline{\underline{\frac{1}{2}}}$$

<u>Nedre integral:</u>

$$\underline{\int_0^1} x\,dx = \sup\{N(\Pi) : \Pi \text{ partisjon}\}$$

$$= \sup\{N(\Pi_n) : n \in \mathbb{N}\}$$

$$\underset{\text{Korollar 8.2.4}}{=} \sup_{n \in \mathbb{N}} \left\{\frac{1}{2}\left(1 - \frac{1}{n}\right)\right\} = \underline{\underline{\frac{1}{2}}}$$

c) Fra b) er f integrerbar siden

$$\underline{\int_0^1} x\,dx = \overline{\int_0^1} x\,dx = \frac{1}{2} = \int_0^1 x\,dx .$$

8.3: 1.) h)

$$\int_1^9 \sqrt{x}^3\,dx = \int_1^9 x^{\frac{3}{2}}\,dx$$

$$= \left[\frac{2}{5}x^{\frac{5}{2}}\right]_{x=1}^{9} = \frac{2}{5}\left(\sqrt{9}^5 - \sqrt{1}^5\right)$$

$$= \underline{\underline{\frac{484}{5}}}$$

3.) c)

$$\int_1^4 \frac{1}{2x+1}\,dx = \left[\ln(2x+1)\frac{1}{2}\right]_{x=1}^{4}$$

$$= \frac{1}{2}\left[\ln(9) - \ln(3)\right] = \frac{1}{2}(2\ln 3 - \ln 3)$$

$9 = 3^2$

$$= \underline{\underline{\frac{\ln 3}{2}}}$$

g)

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(\frac{1}{\cos^2 x} + \frac{1}{e^{7x}}\right)dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{\cos^2 x}\,dx + \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} e^{-7x}\,dx$$

$$= \left[\tan x\right]_{x=-\frac{\pi}{4}}^{\frac{\pi}{4}} + \left[e^{-7x}\left(-\frac{1}{7}\right)\right]_{x=-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$= 1 - (-1) + \left(-\frac{1}{7}\right)\left(e^{-\frac{7\pi}{4}} - e^{\frac{7\pi}{4}}\right)$$

$$= 2 + \frac{e^{\frac{7\pi}{4}} - e^{-\frac{7\pi}{4}}}{7}$$

4.) d) $\displaystyle\int_0^{\pi} \underbrace{\cos x\, e^{\sin x}} dx = \left[e^{\sin x}\right]_{x=0}^{\pi}$

$$= e^{\sin \pi} - e^{\sin 0} = e^0 - e^0 = \underline{\underline{0}}$$

6.) <u>Anta:</u> f kont. og g deriverbar. Def:

$$G(x) = \int_a^{g(x)} f(t)\, dt$$

$G'(x)$?

La $\displaystyle F(u) := \int_a^u f(t)\, dt$

Da er: $G(x) = F(g(x))$

<u>Så:</u> $G'(x) = D[F(g(x))] = F'(g(x))g'(x)$

kjerne-regel

$$= f(g(x))\,\underline{\underline{g'(x)}}$$

Analysens fundamentalteorem

$$\boxed{F'(u) = f(u)}$$

b) i) $D\left[\int_0^{\sin x} \underbrace{t e^{-t}}\, dt\right] \underset{\text{Formel over}}{=} \underbrace{\sin x\, e^{-\sin x}}\ \underline{\cos x}$

$$\left(= \frac{1}{2}\sin(2x)\, e^{-\sin x} \right)$$

$$2\sin x \cos x = \sin 2x$$

iii) $D\left[\int_{\sin x}^{0} \frac{1}{\sqrt{1-t^2}}\, dt\right] = D\left[-\int_0^{\sin x} \frac{1}{\sqrt{1-t^2}}\, dt\right]$

$$\underset{\text{Formel}}{=} -\underbrace{\frac{1}{\sqrt{1-\sin^2 x}}}\cos x = \frac{-\cos x}{\sqrt{\cos^2 x}} = \frac{-\cos x}{\cos x} = \underline{\underline{-1}}$$

$$\boxed{\sin^2 x + \cos^2 x = 1}$$

8.3: 3) e)

$$\int_0^1 \frac{1}{\sqrt{9-x^2}}\,dx = \int_0^1 \frac{1}{\sqrt{9\left(1-\frac{x^2}{9}\right)}}\,dx$$

$$= \int_0^1 \frac{1}{\sqrt{9}\sqrt{1-\frac{x^2}{9}}}\,dx = \frac{1}{3}\int_0^1 \frac{1}{\sqrt{1-\frac{x^2}{9}}}\,dx$$

$$= \frac{1}{3}\int_0^1 \frac{1}{\underbrace{\sqrt{1-\left(\frac{x}{3}\right)^2}}_{y}}\,dx = \frac{1}{3}\Big[\underbrace{3\arcsin\left(\frac{x}{3}\right)}\Big]_{x=0}^{1}$$

$$= \arcsin\left(\frac{1}{3}\right) - \arcsin(0)$$

$$= \underline{\underline{\arcsin\left(\frac{1}{3}\right)}}$$

7.) b)

$$\lim_{x\to\infty} \frac{\int_1^x e^{\frac{1}{t}}\,dt}{x^2} = \lim_{x\to\infty} \frac{D\left[\int_1^x e^{\frac{1}{t}}\,dt\right]}{2x}$$

Teller → ∞

Nevner → ∞

" $\frac{\infty}{\infty}$ " : L'Hôpital

$$= \lim_{x\to\infty} \frac{e^{\frac{1}{x}}}{2x}$$

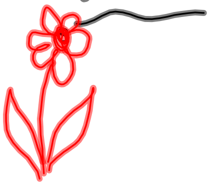
Analysens Fundamentalteorem

$$= \underline{\underline{0}}$$

Teller → $e^0 = 1$

Nevner → ∞

8.4:



1)e) $\int \frac{4}{\sqrt{7-x^2}}\,dx$

$= 4\int \frac{1}{\sqrt{7(1-\frac{x^2}{7})}}\,dx$

$= \underbrace{4\int \frac{1}{\sqrt{7}\sqrt{1-\left(\frac{x}{\sqrt{7}}\right)^2}}\,dx}$ $= 4\,\underbrace{\arcsin\left(\frac{x}{\sqrt{7}}\right)}$

$\underline{\underline{+\,C}}$

2)e) $\int \frac{1+x}{1+x^2}\,dx = \underbrace{\int \frac{1}{1+x^2}\,dx} + \underbrace{\int \frac{x}{1+x^2}\,dx}$

$= \arctan(x) + \underline{\underline{\frac{1}{2}\ln\left(\underline{1+x^2}\right)}} + C$

$\boxed{C_1,\, C_2:\ C = C_1 + C_2}$

3.) b) $\int \sin(2x)\,\frac{e^{\cos^2 x}}{e^{\sin^2 x}}\,dx = \int \sin(2x)\,e^{\cos^2 x - \sin^2 x}\,dx$

$= \int \sin(2x)\,e^{\cos(2x)}\,dx$

$$= -\frac{1}{2}e^{\cos(2x)} + C$$

5.) $f : (0,\infty) \to \mathbb{R}$,

For alle

$(\star)$: $f(xy) = f(x) + f(y) \quad \forall x, y \in (0,\infty)$

$(\star\star)$: f er derivbar i $x = 1$, m/ $f'(1) = k$.

a) $f(1) = 0$? Velg $x = y = 1$:

$$f(1) = f(1\cdot 1) = f(xy) \underset{\star}{=} f(x) + f(y) = f(1) + f(1) = 2f(1)$$

Så $\underline{\underline{f(1) = 0}}$.

b) $f(x+h) = f(x) + f\left(1 + \frac{h}{x}\right)$?

$$f(x) + f\left(1+\frac{h}{x}\right) \underset{(\ast)}{=} f\left(x\left(1+\frac{h}{x}\right)\right) = f(x+h)$$

$f'(x) = \frac{k}{x}$?

$$f'(x) \underset{\text{Hvis eksisterer: Def. av derivert}}{=} \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$\underset{\text{over}}{=} \lim_{h \to 0} \frac{\cancel{f(x)} + f\left(1+\frac{h}{x}\right) - \cancel{f(x)}}{h}$$

$$= \lim_{h\to 0} \frac{f\left(1+\frac{h}{x}\right)}{\frac{xh}{x}} = \frac{1}{x} \lim_{h\to 0} \frac{f\left(1+\frac{h}{x}\right) - 0}{\frac{h}{x}}$$

$$\underset{y := \frac{h}{x},\ h\to 0,\ y\to 0}{=} \frac{1}{x} \lim_{y\to 0} \frac{f(1+y) - f(1)}{y} \underset{\text{Def. av } f'(1)}{=} \frac{1}{x} f'(1) = \underline{\underline{\frac{k}{x}}}$$

c) $f'(x) = \frac{k}{x}$ (b)

$\Downarrow$ (Analysens fundamentalteorem)

$f(x) = k \ln x + C$

Men: $\underset{\overset{\parallel}{0}}{f(1)} = k \ln(1) + C = 0 + C = C$

$\Downarrow$

$C = 0$

Så: $\underline{\underline{f(x) = k \ln x}}$

8.5: 5.) $\lim_{n\to\infty} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{1}{\sqrt{i}}$?

$$\begin{cases} \Pi_n = \{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, 1\} \\ U_n = \{\frac{1}{n}, \frac{2}{n}, \dots, 1\} \quad (\text{samme som i 4.}) \\ f(x) = \frac{1}{\sqrt{x}} \end{cases}$$

$$R_f(\Pi_n, U_n) = \sum_{i=1}^{n} f\left(\frac{i}{n}\right) \frac{1}{n} = \sum_{i=1}^{n} \frac{\sqrt{n}}{\sqrt{i}} \frac{1}{n}$$

$$= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{1}{\sqrt{i}}$$

Fra Korollar 8.5.4:

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{1}{\sqrt{i}} = \int_0^1 \frac{1}{\sqrt{x}}\, dx = \int_0^1 x^{-\frac{1}{2}}\, dx$$

$$= \left[2x^{\frac{1}{2}} \right]_{x=0}^{1} = 2 - 0 = \underline{\underline{2}}$$