Plenum, 13/9-13

3.4: 15 a

3.5: 1 ab, 3a, 5, 9

Midtveis 2012: 1, 2, 3, 13

3.4: 15.) a) $z^3 + iz^2 + z = 0$ (~)

$$z(z^2 + iz + 1) = 0$$

$\underline{z = 0}$, $z^2 + iz + 1 = 0$

$$z = \frac{-i \pm \sqrt{(i)^2 - 4\cdot 1\cdot 1}}{2\cdot 1}$$

$$= \frac{-i \pm \sqrt{-1-4}}{2} = \frac{-i \pm \sqrt{-5}}{2}$$

$$= \underline{\frac{-i \pm \sqrt{5}\, i}{2}}$$

Løsningene er:

$$z = 0, \quad \underline{\underline{z = \frac{-i + \sqrt{5}\, i}{2}}}, \quad z = \frac{-i - \sqrt{5}\, i}{2}$$

3.5: 1.) b) $z^4 - 1 = 0$

$$z^4 = 1 = e^0$$

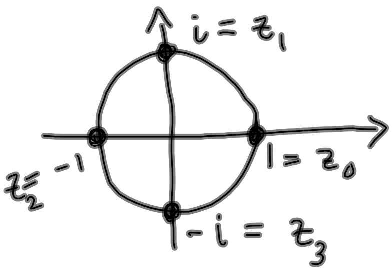
$z_+ = e^{\frac{2\pi}{n} i}$, n er # rötter. $\boxed{n=4}$

$$z_0 = e^{i\frac{0}{4}} = e^0 = 1, \quad z_+ = e^{\frac{2\pi}{4}i} = e^{\frac{\pi}{2}i}$$

$$z_1 = z_+ z_0 = e^{i(0+\frac{\pi}{2})} = e^{\frac{\pi}{2}i} = i$$ (figur el. regning)

$$z_2 = z_+ z_1 = -1$$

$$z_3 = -i$$



Kompleks faktorisering:

$$z^4 - 1 = (z-1)(z-i)(z+1)(z+i)$$

Reell faktorisering: (ganger $(z-i)(z+i)$)

$$z^4 - 1 = (z-1)(z+1)(z^2+1)$$

3) a) $z^4 + 2z^2 + 1$:

$$z^4 + 2z^2 + 1 = 0 \quad (\sim)$$

La $w = z^2$:

$$z^4 + 2z^2 + 1 = (z^2)^2 + 2z^2 + 1 = w^2 + 2w + 1 = 0$$

Annengradsformel:

$$w = \frac{-2 \pm \sqrt{4 - 4\cdot 1\cdot 1}}{2} = -1$$

{ multiplisitet 2

$(w+1)^2 = w^2 + 2w + 1$

$\parallel$

$(w-(-1))(w-(-1))$ }

$z_1^2 = -1 \Rightarrow z_1 = \pm i$

$z_2^2 = -1 \Rightarrow z_2 = \pm i$

Kompleks faktorisering:

$$z^4 + 2z^2 + 1 = (z-i)^2 (z+i)^2$$

Reell faktorisering:

$$z^4 + 2z^2 + 1 = (z^2+1)^2$$

{ $(w-(-1))^2 = (w+1)^2$

$= (z^2+1)^2$ }

5.) a) Vis: i er en rot i

$$P(z) = z^4 + 2z^3 + 4z^2 + 2z + 3$$

$$P(i) = i^4 + 2i^3 + 4i^2 + 2i + 3$$

(Sett $z = i$, se på $P(z) = P(i)$, og vis at dette er 0)

$= \cancel{1} - \cancel{2i} - \cancel{4} + \cancel{2i} + \cancel{3} = \underline{0}$

Så i <u>er en rot</u> i $P(z)$.

b) $\underline{i}$ er en rot i $P(z)$. $P(z)$ er et reelt polynom.
Lemma 3.5.3 at $\underline{-i}$ også er en rot i $P(z)$.

Dermed må $P(z)$ være delelig med $(z-i)(z+i)$
$= \underline{z^2+1}$

$$
\begin{array}{l}
z^4 + 2z^3 + 4z^2 + 2z + 3 \div z^2+1 = \underline{z^2 + 2z + 3} \\
\underline{-(z^4+z^2)} \\
\quad 2z^3 + 3z^2 + 2z + 3 \\
\quad \underline{-(2z^3 + 2z)} \\
\qquad 3z^2 + 3 \\
\qquad \underline{-(3z^2+3)} \\
\qquad\quad 0
\end{array}
$$

Så: $P(z) = (z^2+1)(z^2+2z+3)$

<u>Annengradsregel</u>:

$$z = \frac{-2 \pm \sqrt{4 - 4\cdot 1\cdot 3}}{2\cdot 1} = \frac{-2 \pm \sqrt{-8}}{2} = \frac{-2 \pm \sqrt{8}\, i}{2}$$

$$= \frac{-2 \pm \sqrt{4\cdot 2}\, i}{2} = \frac{-2 \pm 2\sqrt{2}\, i}{2}$$

$$= \underline{-1 \pm \sqrt{2}\, i}$$

Kompleks faktorisering:

$$P(z) = (z - i)(z + i)\big(z - (-1 + \sqrt{2}i)\big)\big(z - (-1 - \sqrt{2}i)\big)$$

$$= (z - i)(z + i)(z + 1 - \sqrt{2}i)(z + 1 + \sqrt{2}i)$$

Reell faktorisering:

$$P(z) = (z^2 + 1)(z^2 + 2z + 3)$$

(sjekk faktorisering ved å gange ut)

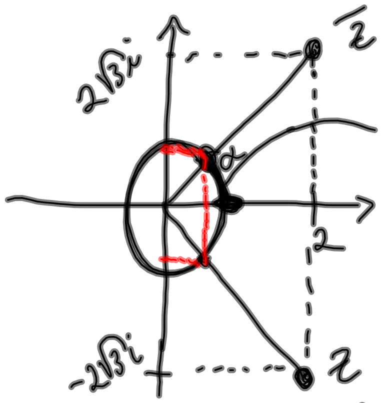
Midtveis 2012

1) $z = 2 - 2\sqrt{3}\, i$:

$$r = |z| = \sqrt{2^2 + (-2\sqrt{3})^2} = \sqrt{4 + 4 \cdot 3} = \sqrt{16} = \underline{4}$$

$$\left.\begin{aligned} 4\cos\theta &= 2 \\ 4\sin\theta &= -2\sqrt{3} \end{aligned}\right\} \Rightarrow \left.\begin{aligned} \cos\theta &= \tfrac{1}{2} \\ \sin\theta &= -\tfrac{\sqrt{3}}{2} \end{aligned}\right\} \Rightarrow \theta = \underline{\frac{5\pi}{3}}$$

figur



$$\cos\alpha = \frac{1}{2}, \quad \sin\alpha = \frac{\sqrt{3}}{2}$$

$$\Downarrow$$

$$\alpha = \frac{\pi}{3}$$

$$\theta = 2\pi - \alpha = 2\pi - \frac{\pi}{3} = \frac{6\pi - \pi}{3} = \frac{5\pi}{3}$$

$$z = \underline{4e^{i\frac{5\pi}{3}}}$$

$\boxed{C}$

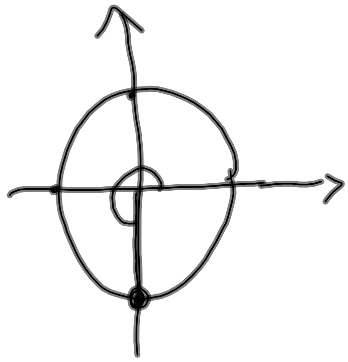
(2) $r = \sqrt{2},\ \theta = \frac{7\pi}{2}$:

$$z = \sqrt{2}\left(\cos\frac{7\pi}{2} + i\sin\frac{7\pi}{2}\right)$$

Merk: $\frac{7\pi}{2} = \frac{4\pi + 3\pi}{2} = 2\pi + \frac{3\pi}{2}$

$$\Rightarrow \cos\frac{7\pi}{2} = \cos\left(2\pi + \frac{3\pi}{2}\right) = \cos\frac{3\pi}{2}$$

$$\sin\frac{7\pi}{2} = \sin\frac{3\pi}{2}$$



Så: $z = \sqrt{2}\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$

$= \sqrt{2}\,(0 + i(-1)) = \underline{-\sqrt{2}\,i}$

$\boxed{A}$

③ $P(z) = z^3 - 4z^2 + 5z$:

Kan: sette inn tallene direkte $\rightsquigarrow$ Tar lang tid!

Smartere: $P(z) = z(z^2 - 4z + 5)$

Nå: Kan sette alle tallene inn i $z^2 - 4z + 5$

eller (lurere) løse $z^2 - 4z + 5 = 0$.

$$z = \frac{4 \pm \sqrt{16 - 20}}{2} = \frac{4 \pm \sqrt{-4}}{2} = \frac{4 \pm \sqrt{4}\,i}{2}$$

$$= \frac{4 \pm 2i}{2} = \underline{2 \pm i} \quad \boxed{B}$$

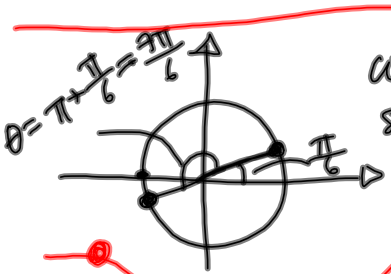
13) 3. rot til $z = -4\sqrt{3} - 4i$:

$$r = |z| = \sqrt{(-4\sqrt{3})^2 + (-4)^2} = \sqrt{16 \cdot 3 + 16}$$
$$= \sqrt{4 \cdot 16} = 2 \cdot 4 = \underline{\underline{8}}$$

Modulusen til 3. røttene til z er $(8)^{\frac{1}{3}} = 2$
(fordi $2^3 = 4 \cdot 2 = 8$). Altså kan $\boxed{B}$ og $\boxed{E}$ stemme.

Hva av $\boxed{B}$ og $\boxed{E}$ er rett?

$$8\cos\theta = -4\sqrt{3} \Rightarrow \cos\theta = -\frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{7\pi}{6}$$
$$8\sin\theta = -4 \qquad \sin\theta = -\frac{1}{2}$$



$\cos\alpha = \frac{\sqrt{3}}{2}$
$\sin\alpha = \frac{1}{2} \Rightarrow \alpha = \frac{\pi}{6}$

Så argumentet til første 3. rot av z er:

$$\frac{\theta}{3} = \frac{\frac{7\pi}{6}}{3} = \underline{\frac{7\pi}{18}}$$

$\boxed{E}$