Plenum 15/11-13

9.2: 1efgh, 3 ,c,d, 7, 9, 15, 23, 25

9.3: 1d, 3abe, 5afg, 9, 17, 21, (25), 27, 31

9.5: 1ab, 3ac, 6, 10

9.2: 1) h) $\int \arcsin\sqrt{x}\,dx = \int \arcsin(u)\,2u\,du$

$u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}}dx$

$2\sqrt{x}\,du = dx$

$2u\,du = dx$

$= 2\int u \arcsin(u)\,du$

$= 2\left[\frac{1}{2}u^2\arcsin(u) - \int \frac{1}{2}u^2 \frac{1}{\sqrt{1-u^2}}du\right]$

$v' = u$

$w = \arcsin(u)$

$v = \frac{1}{2}u^2$

$w' = \frac{1}{\sqrt{1-u^2}}$

M: $\int \frac{-u^2}{\sqrt{1-u^2}}du$

$= \int \frac{1-u^2-1}{\sqrt{1-u^2}}du = \int \frac{1-u^2}{\sqrt{1-u^2}}du - \int \frac{1}{\sqrt{1-u^2}}du$

$= \int \sqrt{1-u^2}\,du - \arcsin(u)$

$$= \int \sqrt{1-\sin^2 v}\,\cos v\,dv - \arcsin(u)$$

$u = \sin v$
$du = \cos v\,dv$

$$= \int \sqrt{\cos^2 v}\,\cos v\,dv - \arcsin(u)$$

$$= \int \cos^2 v\,dv - \arcsin(u)$$

M2: $\int \cos^2 v\,dv = \int \frac{\cos(2v)+1}{2}\,dv$

$$= \frac{1}{2}\left[\frac{1}{2}\sin(2v) + v\right] + C$$

$$= \frac{1}{4}\sin(2v) + \frac{1}{2}v + C$$

M: $\int \frac{-u^2}{\sqrt{1-u^2}}\,du = \frac{1}{4}\sin(2\arcsin(u)) - \frac{1}{2}\arcsin(u) + C$

Så: $\int \arcsin(\sqrt{x})\,dx = u^2\arcsin(u) + \frac{1}{4}\sin(2\arcsin(u)) - \frac{1}{2}\arcsin(u) + C$

$\sqrt{x} = u$

$$= x\arcsin(\sqrt{x}) + \frac{1}{4}\sin(2\arcsin(\sqrt{x})) - \frac{1}{2}\arcsin(\sqrt{x}) + C$$

3.) c) $$\int_4^9 \frac{\sqrt{x}+1}{1-\sqrt{x}}\,dx = \int_2^3 \frac{u+1}{1-u}\,2u\,du$$

$u=\sqrt{x}$
$du=\frac{1}{2\sqrt{x}}dx$
$2u\,du=dx$
$x=4 \Rightarrow u=2$
$x=9 \Rightarrow u=3$

$$=-2\int_2^3 \frac{u^2+u}{u-1}\,du$$

M: Polynomdiv.

$$u^2+u : u-1 = u+2 + \frac{2}{u-1}$$

$$\begin{array}{l} -(u^2-u) \\ \hline 2u \\ -(2u-2) \\ \hline 2 \end{array}$$

$$-2\int_2^3 \frac{u^2+u}{u-1}\,du = -2\int_2^3 \left\{u+2+\frac{2}{u-1}\right\}du$$

$$=-2\left[\tfrac{1}{2}u^2+2u+2\ln|u-1|\right]_{u=2}^3$$

$$=-2\left(\tfrac{1}{2}9+2\cdot 3+2\ln 2-\tfrac{1}{2}2^2-4-2\ln 1\right)$$

$$=\underline{\underline{-4\ln 2-9}}$$

d) $\int_0^3 \arctan\sqrt{x}\,dx = \int_0^{\sqrt{3}} \arctan(u)\,2u\,du$

$x=0 \Rightarrow u=0$

$x=3 \Rightarrow u=\sqrt{3}$

$u=\sqrt{x}$

$du = \frac{1}{2\sqrt{x}}\,dx$

$2u\,du = dx$

$v = \arctan u$

$w' = u$

$v' = \frac{1}{1+u^2}$

$w = \frac{1}{2}u^2$

$= 2\int_0^{\sqrt{3}} \arctan(u)\,u\,du = 2\left(\left[\frac{1}{2}u^2 \arctan u\right]_{u=0}^{\sqrt{3}} - \int_0^{\sqrt{x}} \frac{\frac{1}{2}u^2}{1+u^2}\,du\right) = 3\arctan\sqrt{3} - \int_0^{\sqrt{3}} \frac{1+u^2-1}{1+u^2}\,du$

$= 3\arctan\sqrt{3} - \int_0^{\sqrt{3}} 1\,du + \int_0^{\sqrt{3}} \frac{1}{1+u^2}\,du$

$= 3\arctan\sqrt{3} - \sqrt{3} + \left[\arctan u\right]_{u=0}^{\sqrt{3}}$

$= 4\arctan\sqrt{3} - \sqrt{3}$

$= \frac{4\pi}{3} - \sqrt{3}$

15.) $$\int_0^{\sqrt{3}} \frac{1+x}{\sqrt{4-x^2}}\,dx = \int_0^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\,dx + \int_0^{\sqrt{3}} \frac{x}{\sqrt{4-x^2}}\,dx$$

$$= \int_0^{\sqrt{3}} \frac{1}{\sqrt{4\left(1-\left(\frac{x}{2}\right)^2\right)}}\,dx + \int_4^1 \left(-\frac{1}{2}\right)\frac{1}{\sqrt{u}}\,du$$

$u = 4 - x^2$

$du = -2x\,dx$

$$\int \frac{x}{\sqrt{4-x^2}}\,dx = \int \frac{\cancel{x}}{\sqrt{u}}\left(-\frac{1}{2\cancel{x}}\right)du$$

$x = 0 \Rightarrow u = 4$

$x = \sqrt{3} \Rightarrow u = 1$

$$= \frac{1}{2}\int_0^{\sqrt{3}} \frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}\,dx + \frac{1}{2}\int_1^4 u^{-\frac{1}{2}}\,du$$

$$= \frac{1}{\cancel{2}}\left[\arcsin\left(\frac{x}{2}\right)\cancel{2}\right]_{x=0}^{\sqrt{3}} + \frac{1}{\cancel{2}}\left[\cancel{2}\,u^{\frac{1}{2}}\right]_{u=1}^{4}$$

$$= \arcsin\frac{\sqrt{3}}{2} - \arcsin 0 + 2 - 1 = \underline{\underline{\frac{\pi}{3} + 1}}$$

9.3: 5.) a) $\int \frac{x^2+2x-3}{x+1}\,dx$

Polynomdivisjon:

$$x^2+2x-3 : x+1 = x+1-\frac{4}{x+1}$$
$$-(x^2+x)$$
$$x-3$$
$$-(x+1)$$
$$-4$$

$$\int \frac{x^2+2x-3}{x+1}\,dx = \int \left\{x+1-\frac{4}{x+1}\right\}dx$$
$$= \frac{1}{2}x^2 + x - 4\ln|x+1| + C$$

9.) $\int \frac{x+1}{(x-1)(x^2+x+1)}\,dx$

$\hookrightarrow x = \frac{-1 \pm \sqrt{1-4}}{2}$: Kan ikke faktoriseres mer reelt.

Delbrøkoppspaltning:

$$\frac{x+1}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$$

$$x+1 = A(x^2+x+1) + (Bx+C)(x-1)$$
$$= x^2(A+B) + x(A-B+C) + (A-C)$$

$A+B=0$, $A-B+C=1$, $A-C=1$

$$A = -B \rightsquigarrow -2B + C = 1$$

$$\Downarrow$$

$$C = 1 + 2B$$

<u>3. lign:</u> $-B - 1 - 2B = 1 \Rightarrow -3B = 2, \quad B = -\frac{2}{3}$

$$\Rightarrow A = \frac{2}{3}, \quad C = -\frac{1}{3}$$

$$\int \frac{x+1}{(x-1)(x^2+x+1)}\,dx = \frac{2}{3}\int \frac{1}{x-1}\,dx - \frac{1}{3}\int \frac{2x+1}{x^2+x+1}\,dx$$

$$= \frac{2}{3}\ln|x-1| - \frac{1}{3}\int \frac{1}{u}\,du$$

$u = x^2+x+1$
$du = (2x+1)\,dx$

$$= \frac{2}{3}\ln|x-1| - \frac{1}{3}\ln|x^2+x+1| + C$$

$$= \frac{1}{3}\ln\left[\frac{(x-1)^2}{x^2+x+1}\right] + C$$

17.) $\int \frac{1}{x^3+8}\,dx = \int \frac{1}{(x+2)\underbrace{(x^2-2x+4)}}\,dx$

Kan ikke faktoriseres mer reelt fra 2. gradsformel

-2 er en rot i x^3+8 $\left((-2)^3+8=0\right)$

Polynomdiv: $(x^3+8):(x+2)=x^2-2x+4$

$$\begin{array}{l} -(x^3+2x^2) \\ \hline -2x^2+8 \\ -(-2x^2-4x) \\ \hline 4x+8 \\ -(4x+8) \\ \hline 0 \end{array}$$

Delbrøksoppspaltning:

$$\frac{1}{(x+2)(x^2-2x+4)} = \frac{A}{x+2} + \frac{Bx+C}{x^2-2x+4}$$

$$\begin{cases} 1 = A(x^2-2x+4)+(Bx+C)(x+2) \\ \quad = x^2(A+B)+x(-2A+2B+C)+(4A+2C) \end{cases}$$

$A+B=0$, $-2A+2B+C=0$, $4A+2C=1$

$A=-B \Rightarrow 2B+2B+C=0$

$C=-4B \rightarrow -4B-8B=1$

$\underline{B=-\frac{1}{12}}$

$\underline{C=\frac{1}{3}}$, $\underline{A=\frac{1}{12}}$

$$\int \frac{1}{x^3+8}\,dx = \frac{1}{12}\int \frac{1}{x+2}\,dx + \int \frac{-\frac{1}{12}x+\frac{1}{3}}{x^2-2x+4}\,dx$$

$$= \frac{1}{12}\ln|x+2| - \frac{1}{24}\int \frac{2x-8}{x^2-2x+4}\,dx$$

$$= \frac{1}{12}\ln|x+2| - \frac{1}{24}\left(\int \frac{2x-2}{x^2-2x+4}\,dx + \int \frac{-6}{x^2-2x+4}\,dx\right)$$

$$= \frac{1}{12}\ln|x+2| - \frac{1}{24}\int \frac{1}{u}\,du + \frac{1}{4}\int \frac{1}{x^2-2x+4}\,dx$$

$u = x^2-2x+4$
$du = (2x-2)\,dx$

$$= \frac{1}{12}\ln|x+2| - \frac{1}{24}\ln|x^2-2x+4| + \frac{1}{4}\int \frac{1}{(x-1)^2+3}\,dx$$

$$= \frac{1}{12}\ln|x+2| - \frac{1}{24}\ln|x^2-2x+4| + \frac{1}{4}\int \frac{1}{3\left[\left(\frac{x-1}{\sqrt{3}}\right)^2+1\right]}\,dx$$

$$= \frac{1}{12}\ln|x+2| - \frac{1}{24}\ln|x^2-2x+4| + \frac{\sqrt{3}}{12}\arctan\left(\frac{x-1}{\sqrt{3}}\right) + C$$

9.5: 6.) $\underbrace{\int_0^1 \ln(x^3+x^2)\,dx}$ konv. eller div.?

$$\int_0^1 \ln(x^3+x^2)\,dx = \lim_{a\to 0^+} \int_a^1 \ln(x^3+x^2)\,dx$$

Merk: Har ingen problemer med at int. $\to +\infty$, trenger derfor kun å begrense nedenfra.

$$\ln(x^3) < \ln(x^3+x^2)$$

siden ln er voksende

NB: $\int_a^1 \ln(x^3)\,dx = 3\left[x\ln x - x\right]_{x=a}^{1}$

Delvis int: $\int \ln(x^3)\,dx = \int 3\ln x\,dx = 3\int 1\cdot \ln x\,dx$

$$= -3 - a\ln a + a$$

"$\frac{\infty}{\infty}$": L'Hopital

M: $\lim_{a\to 0^+} a\ln a = \lim_{a\to 0^+} \frac{\ln a}{\frac{1}{a}} = \lim_{a\to 0^+} \frac{\frac{1}{a}}{-\frac{1}{a^2}}$

$$= \lim_{a\to 0^+} -a = 0$$

Så: $\lim_{a\to 0^+} \int_a^1 \ln(x^3)\,dx = -3$

Siden $\ln(x^3) < \ln(x^3+x^2)$ og $\int_0^1 \ln(x^3)\,dx$ konvergerer, gir sammenligningskriteriet at $\int_0^1 \ln(x^3+x^2)\,dx$ konvergerer.

10.) For hvilke p konv.

$$\underbrace{\int_0^{\frac{1}{2}} \frac{1}{x\,|\ln x|^p}\,dx}\ ?$$

$$\int_0^{\frac{1}{2}} \frac{1}{x\,|\ln x|^p}\,dx = \int_{-\infty}^{-\ln 2} \frac{1}{|u|^p}\,du = \int_{\ln 2}^{\infty} \frac{1}{u^p}\,du$$

$u = \ln x$

$du = \frac{1}{x}dx$

$x = 0 \rightsquigarrow u = -\infty$

$x = \frac{1}{2} \rightsquigarrow u = \ln\frac{1}{2} = -\ln 2$

ved symmetri

$$= \underbrace{\int_{\ln 2}^{1} \frac{1}{u^p}\,du}_{\text{Konvergerer for alle } p} + \underbrace{\int_1^{\infty} \frac{1}{u^p}\,du}_{\text{Set. 9.5.4: Konv. for } p>1,\ \text{Div. for } p\le 1}$$

Så: $\int_0^{\frac{1}{2}} \frac{1}{x\,|\ln x|^p}\,dx$ konv. for $p>1$ og divergere for $p \le 1$.