Plenum 25/10-13

7.1: 1, 5, 7, 8, 15

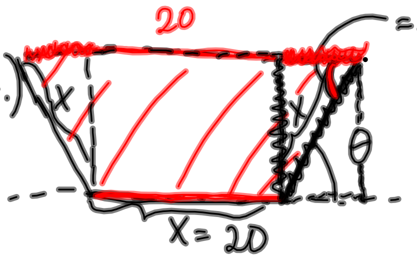
7.2: 1, 3, (5), 7, 9, 13

7.4: 1ab(c), 3, 5, (9), 10, 8

7.5: 3ab

7.6: 1a(e)h, 2ab, 3a(b)e, 5, 7

7.1: 7.)

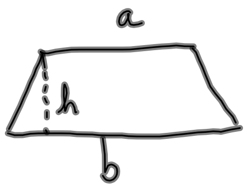


60 cm; $3x = 60$

$x = 20$

Hvilken θ max'er areal av trapes?

Areal trapes:



$$\frac{(a+b)h}{2}$$

I arealformel: $a = 20$, $b = 20 + 2 \cdot 20\cos\theta$, $h = 20\sin\theta$

$$\underbrace{A(\theta)}_{\text{areal}} = \frac{(20 + 20 + 40\cos\theta)\,20\sin\theta}{2}$$

$$= 400(\sin\theta + \cos\theta\sin\theta)$$

$$A'(\theta) = 400\left(\cos\theta + (-\sin\theta)\sin\theta + \cos\theta\cos\theta\right)$$

$$= 400\left(\cos\theta - \sin^2\theta + \cos^2\theta\right)$$

$$= 400\left(\cos\theta - (1-\cos^2\theta) + \cos^2\theta\right)$$

$$= 400\left(2\cos^2\theta + \cos\theta - 1\right) = 0$$

$$\Updownarrow$$

$$2\cos^2\theta + \underbrace{\cos\theta}_{:=y} - 1 = 0$$

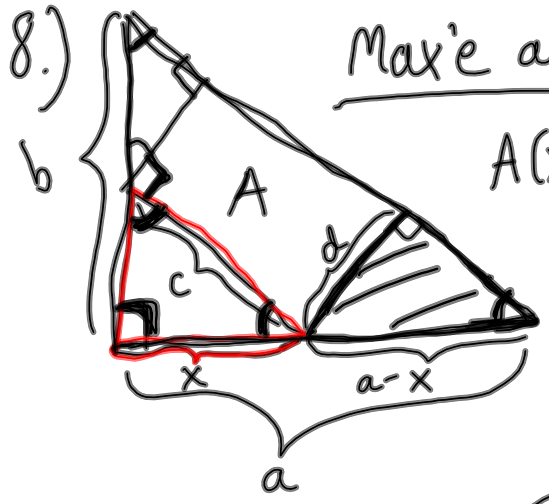
Så: $2y^2 + y - 1 = 0$

$$y = \frac{-1 \pm \sqrt{1+8}}{4} = \frac{-1 \pm 3}{4} = \begin{cases} \frac{1}{2} \\ -1 \end{cases}$$

$y = -1 \Leftrightarrow \cos\theta = -1 \Leftrightarrow \theta = \pi\ (+2k\pi)$. Dette kan ikke maksimere arealet (se fig., el. sett inn i $A(\theta)$, og se at får mindre verdi enn i $y = \frac{1}{2}$ tilfellet).

Det betyr at $y = \cos\theta = \frac{1}{2}$ gir maksimalt areal av trapeset. Dvs. $\theta = \frac{\pi}{3}$ gir max. areal.

8.)



Max'e areal av A?

$$A(x) = c \cdot d$$

$$\frac{c}{x} = \frac{\sqrt{a^2+b^2}}{a}$$

$\downarrow$

Formlikhet stor trekant og liten trekant

$\downarrow$

$$c = \frac{x}{a}\sqrt{a^2+b^2}$$

$$\frac{d}{a-x} = \frac{b}{\sqrt{a^2+b^2}}$$ (Formlikhet mellom liten trekant og stor trekant)

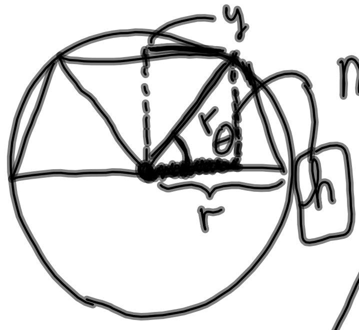
$$d = (a-x)\frac{b}{\sqrt{a^2+b^2}}$$

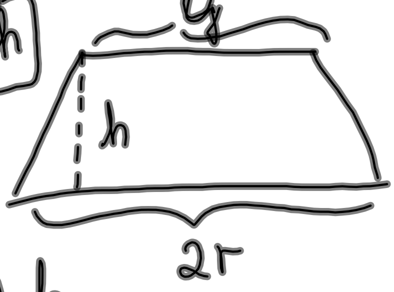
$$A(x) = \frac{(a-x)x\,b}{a}$$

$$A'(x) = \frac{b}{a}(a-2x) = 0$$

$$x = \frac{a}{2}$$ (gjør arealet størst mulig)

15.) max Areal av trapes?





$$A = \frac{(2r+y)h}{2}, \quad h = r\sin\theta,$$

$$\frac{y}{2} = r\cos\theta$$

$$A(\theta) = \frac{(2r + 2r\cos\theta)\, r\sin\theta}{2} = r^2 \sin\theta(1+\cos\theta)$$

$$A'(\theta) = r^2(2\cos^2\theta + \cos\theta - 1) = 0$$

$$\boxed{\cos^2\theta + \sin^2\theta = 1}$$

$$2\cos^2\theta + \underbrace{\cos\theta}_{:=y} - 1 = 0 \rightsquigarrow 2y^2 + y - 1 = 0$$

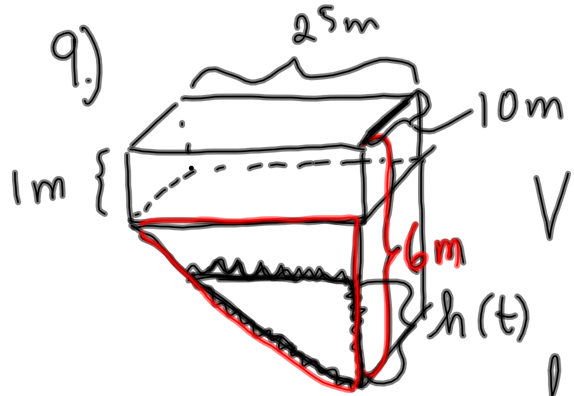
$$\cos\theta = y = \frac{-1 \pm 3}{4} = \begin{cases} \frac{1}{2} \\ -1 \end{cases}$$

$\cos\theta = -1 \Rightarrow \theta = \pi$; kan ikke være optimalt (se fig, el. innsett i $A(\theta)$)

$\cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$ max'er arealet. Så:

$$A^* = A\left(\frac{\pi}{3}\right) = \frac{3r^2\sqrt{3}}{4}$$

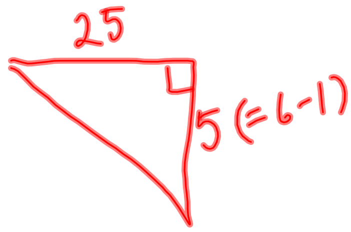
7.2: 9)



Fylles m. 2000 l/min

V(t) = vannvolum ved tid t

h(t) = vannhøyde i dyp ende ved tid t



Pga. formlikhet med rød trekant: $\frac{25}{5} = 5$

$$V(t) = \frac{h(t) \cdot 5h(t) \cdot 10}{2} = \frac{50}{2} h^2(t)$$

$$V'(t) = 50\, h(t)\, h'(t)$$ (Deriver på begge sider!)

Når vannstanden er 3 m (kaller denne tiden $\tilde{t}$)

$$2 = 50 \cdot 3 \cdot h'(\tilde{t})$$

$V'(\tilde{t}) = 2\,m^3 = 2000\,l$

$$h'(\tilde{t}) = \frac{1}{75}\ \text{m/min} = \frac{4}{3}\ \text{cm/min}$$

7.4: 5.) $f(x) = \tan(2x)$, injektiv på $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$

$f'(x) = \dfrac{2}{\cos^2(2x)} > 0$ for $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$, så dermed er f strengt voksende og derfor injektiv.

Finn $(f^{-1})'(1)$: Fra Teorem 7.4.6;

$$\left\{ (f^{-1})'(1) = \frac{1}{f'(x)} \right.$$ der x bestemmes fra $f(x) = 1$.

$f(x) = \tan 2x = 1 \Rightarrow x = \frac{\pi}{8}$ (siden $\sin\frac{\pi}{4} = \cos\frac{\pi}{4}$)

Da er:
$$(f^{-1})'(1) = \frac{1}{\frac{2}{\cos^2\left(\frac{\pi}{4}\right)}} = \frac{\left(\frac{\sqrt{2}}{2}\right)^2}{2} = \frac{\cancel{2}}{\cancel{2}\cdot 4} = \underline{\underline{\frac{1}{4}}}$$

8.) Anta: g er omvendt funk. av str. monot. funk. f, som er 2× deriv. bar.

Vis: g er 2× deriverbar i x og

$$g''(x) = -\frac{f''(g(x))\,g'(x)}{f'(g(x))^2}$$

Bevis: Siden f er kont., strengt monoton og deriverbar i $g(x) = y$ med $f'(g(x)) \neq 0$, så gir Teorem 7.4.6 at g er deriverbar i punktet $f(g(x)) = f(f^{-1}(x)) = x$, og at

$$g'(x) = \frac{1}{f'(g(x))} \quad (*)$$

Men da, siden f' er deriverbar, så er HS deriverbar. Dermed er VS deriverbar. Så:

D.v.s. g er 2× deriverbar

$$g''(x) = \left(\frac{1}{f'(g(x))}\right)' = -\frac{1}{[f'(g(x))]^2} f''(g(x)) \cdot g'(x)$$

$$= -\frac{f''(g(x))\,g'(x)}{(f'(g(x)))^2}$$

7.5: $\cot x = \frac{\cos x}{\sin x} = \frac{1}{\tan x}$

$\frac{x}{\tan x}$

3) a) $\lim_{x \to 0} x \cot x = \lim_{x \to 0} x \frac{\cos x}{\sin x}$ $\quad x \to 0$, $\cos x \to 1$, $\sin x \to 0$

$\left\{ \frac{x \cos x}{\sin x} \right.$

$= \lim_{x \to 0} \frac{\frac{\cos x}{\sin x}}{\frac{1}{x}} = \lim_{x \to 0} \frac{-\frac{1}{\sin^2 x}}{-\frac{1}{x^2}}$

"$0 \cdot \infty$": Må omforme til L'Hôpital brøk

"$\frac{\infty}{\infty}$": L'Hôpital

$= \lim_{x \to 0} \left(\frac{x}{\sin x}\right)^2$

"$\frac{0}{0}$": L'Hôpital

M: $\lim_{x \to 0} \frac{x}{\sin x} = \lim_{x \to 0} \frac{1}{\cos x} = \underline{1}$

Så: $\lim_{x \to 0} x \cot x = \lim_{x \to 0} \left(\frac{x}{\sin x}\right)^2 = \underline{\underline{1}}$