Plenum, 6/9-13

3.2: 15

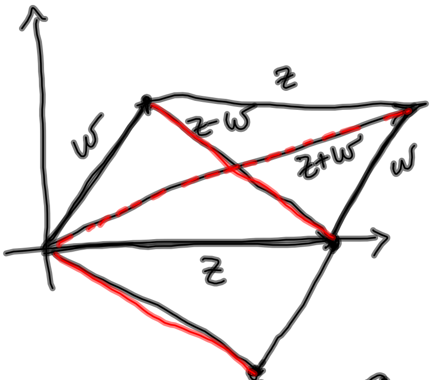
3.3: 1a, 3a, 7, 8, 10

3.4: 1b, d, 3a, 8, 9a, 11d, 15 b

3.5: 1ab, 3a, 5, 9

3.2: 15.) Vis: $|z+w|^2 + |z-w|^2 = 2|z|^2 + 2|w|^2$

Bevis: $|z+w|^2 + |z-w|^2 = (z+w)\overline{(z+w)} + (z-w)\overline{(z-w)}$



$= (z+w)(\bar z + \bar w) + (z-w)(\bar z - \bar w)$

$= z\bar z + \cancel{z\bar w} + \cancel{w\bar z} + w\bar w$
$+ z\bar z - \cancel{z\bar w} - \cancel{w\bar z} + w\bar w$

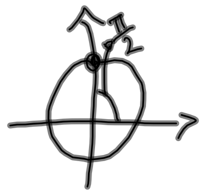
$= 2|z|^2 + 2|w|^2$ ▨

$|z|^2 + |w|^2 + |z|^2 + |w|^2 = 2|z|^2 + 2|w|^2$

$|z-w|^2 + |z+w|^2$

3.3:

1a) $e^{i\frac{\pi}{2}} = e^{0+i\frac{\pi}{2}} = e^{0}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$



$= 0 + i1 = \underline{\underline{i}}$

3a) $\underline{1 + i\sqrt{3}}$

$r = \sqrt{1+3} = \underline{2}$

$1 = 2\cos\theta$
$\sqrt{3} = 2\sin\theta$
$\Rightarrow$ $\cos\theta = \frac{1}{2}$, $\sin\theta = \frac{\sqrt{3}}{2}$ $\Rightarrow \theta = \underline{\frac{\pi}{3}}$

Så: $1 + i\sqrt{3} = 2\underline{\underline{e}}^{i\frac{\pi}{3}}$

7.) De Moivres formel for $n=4$:

$(\cos\theta + i\sin\theta)^4 = \cos(4\theta) + i\sin(4\theta)$

$(\cos\theta + i\sin\theta)^4 = \cos^4\theta + 4\cos^3\theta\, i\sin\theta - 6\cos^2\theta\sin^2\theta - 4\cos\theta\, i\sin^3\theta + \sin^4\theta$

binomialformel

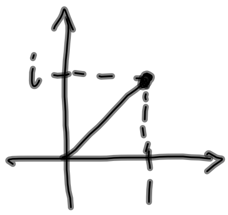
Sml. realledd & imaginærledd

$\cos(4\theta) = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$

$\sin(4\theta) = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$

8.) $(1+i)^{804}$:

$1+i$: $r=\sqrt{1+1}=\sqrt{2}$, $\theta=\frac{\pi}{4}$



$$1+i=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$$

$$(1+i)^{804}=\left(\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\right)^{804}$$

$$=\sqrt{2}^{804}\left(\cos\left(804\frac{\pi}{4}\right)+i\sin\left(804\frac{\pi}{4}\right)\right)$$

De Moivres formel

$$=\left(\sqrt{2}^{2}\right)^{402}\left(\cos(201\pi)+i\sin(201\pi)\right)$$

$$=2^{402}\left(\cos(100\cdot 2\pi+\pi)+i\sin(100\cdot 2\pi+\pi)\right)$$

$$=2^{402}\left(\cos(\pi)+i\sin(\pi)\right)$$

$$=2^{402}(-1+0)=\underline{\underline{-2^{402}}}$$

10.) Vis:

$$\sin(z+w)=\sin z\cos w+\cos z\sin w$$

$$\cos(z+w)=\cos z\cos w-\sin z\sin w$$

La z og w være to komplekse tall.

Påstand: $\sin(z+w)=\sin z\cos w+\cos z\sin w$

HS: $\sin z \cos w + \cos z \sin w$

$$= \frac{e^{iz}-e^{-iz}}{2i}\,\frac{e^{iw}+e^{-iw}}{2} + \frac{e^{iz}+e^{-iz}}{2}\,\frac{e^{iw}-e^{-iw}}{2i}$$

Def. av sin og cos for komplekse tall

$$= \frac{(e^{iz}-e^{-iz})(e^{iw}+e^{-iw})+(e^{iz}+e^{-iz})(e^{iw}-e^{-iw})}{4i}$$

$$= \frac{e^{i(z+w)} + \cancel{e^{i(z-w)}} - \cancel{e^{i(w-z)}} - e^{-i(z+w)} + e^{i(z+w)} - \cancel{e^{i(z-w)}} + \cancel{e^{i(w-z)}} - e^{-i(z+w)}}{4i}$$

$$= \frac{2(e^{i(z+w)}-e^{-i(z+w)})}{4i} = \frac{e^{i(z+w)}-e^{-i(z+w)}}{2i}$$

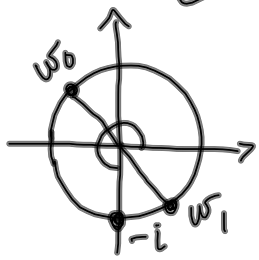
VS: $\sin(z+w) = \dfrac{e^{i(z+w)}-e^{-i(z+w)}}{2i}$

Def. av sin & cos for komplekse tall

Ser at HS = VS, og dermed er påstanden bevist.

3.4:

1) b) $z = -i$:



$r = |z| = 1$, $\theta = \frac{3\pi}{2}$, $z = \underline{e^{i\frac{3\pi}{2}}}$

$$w_0 = z^{\frac{1}{2}} = \underline{\underline{e^{i\frac{3\pi}{4}}}} = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}$$

$$= -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$$

Får neste rot v/å gange w_0 med

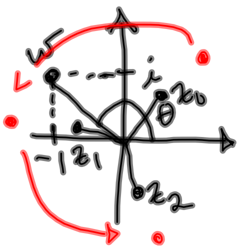
$$e^{i\frac{2\pi}{2}} = e^{i\pi} = w_+ .$$

$$w_1 = e^{i\frac{3\pi}{4}} e^{i\pi} = \underline{\underline{e^{i\frac{7\pi}{4}}}} = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$$

8.) a) $z^3 = -1 + i$

Finn alle 3de røtter til $w = -1 + i$:

$w = -1 + i \;;\; r = |w| = \sqrt{1+1} = \underline{\sqrt{2}}$



$\theta = \underline{\frac{3\pi}{4}}$

$w = \underline{\sqrt{2}\, e^{\frac{3\pi}{4} i}}$

$z_0 = w^{\frac{1}{3}} = \left(\sqrt{2}\, e^{\frac{3\pi}{4} i}\right)^{\frac{1}{3}} = (\sqrt{2})^{\frac{1}{3}} \left(e^{\frac{3\pi}{4} i}\right)^{\frac{1}{3}}$

$= \underline{\underline{2^{\frac{1}{6}} e^{\frac{\pi}{4} i}}}$

$z_+ = \underline{e^{i\frac{2\pi}{3}}} \qquad \left(\begin{matrix} w_+ = e^{\frac{2\pi}{n} i} \\ n = 3 \end{matrix} \right)$

$z_1 = z_0 z_+ = 2^{\frac{1}{6}} e^{\frac{\pi}{4} i} e^{\frac{2\pi}{3} i} = 2^{\frac{1}{6}} e^{\left(\frac{\pi}{4} + \frac{2\pi}{3}\right) i}$

$= 2^{\frac{1}{6}} e^{\left(\frac{3\pi}{12} + \frac{8\pi}{12}\right) i} = \underline{\underline{2^{\frac{1}{6}} e^{\frac{11\pi}{12} i}}}$

$z_2 = z_1 z_+ = \underline{\underline{2^{\frac{1}{6}} e^{i\frac{19\pi}{12}}}}$

$\left.\begin{matrix} z_0^3 \\ z_1^3 \\ z_2^3 \end{matrix}\right\} = -1 + i$

b) $w = 2^{\frac{1}{6}} e^{\frac{11\pi}{12} i}$

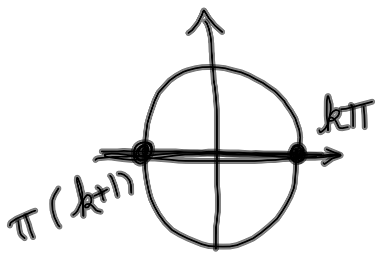
$$w^n = \left(2^{\frac{1}{6}} e^{\frac{11\pi}{12} i}\right)^n = \underline{2^{\frac{n}{6}}}\,\underline{e^{\frac{11\pi n}{12} i}}$$

For at w^n skal være reelt, må

$e^{\frac{11\pi n}{12} i}$ være rellt, dvs. at

$\frac{11\pi n}{12}$ må være lik $k\pi$ for en eller annen

$k \in \mathbb{Z}$.



$$\frac{11\pi n}{12} = k\pi$$

$$11\cancel{\pi} n = 12k\cancel{\pi}$$

$$n = \frac{12k}{11} \qquad (\sim)$$

Må velge k så liten som mulig s.a. n er et pos. naturlig tall. 11 (nevneren i $(\sim)$) er et primtall, så vi må ha $k = 11$.

$$n = \frac{12 \cdot \cancel{11}}{\cancel{11}} = \underline{\underline{12}}$$

9) a) $\underset{a}{x^2} + \overset{b}{2}x + \overset{c}{\underline{4}} = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{4 - 4 \cdot 1 \cdot 4}}{2 \cdot 1}$$

$$= \frac{-2 \pm \sqrt{-\underline{12}}}{2} = \frac{-2 \pm \sqrt{12}\, i}{2}$$

$$= \frac{-2 \pm \sqrt{4 \cdot 3}\, i}{2} = \frac{-2 \pm 2\sqrt{3}\, i}{2}$$

$$= \underline{\underline{-1 \pm \sqrt{3}\, i}}$$

11.) d) $z^2 + (1 - i)z \overset{c}{-i} = 0$

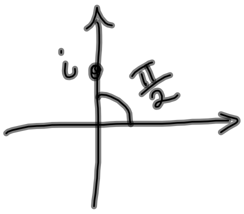
$a = 1, \; b = 1 - i, \; c = -i$

$$z = \frac{-(1-i) \pm \sqrt{(1-i)^2 - 4 \cdot 1 \cdot (-i)}}{2 \cdot 1}$$

$$= \frac{-1 + i \pm \sqrt{\cancel{1} - 2i - \cancel{1} + 4i}}{2} = \frac{-1 + i \pm \sqrt{2i}}{2}$$

$$= \frac{-1 + i \pm \sqrt{2}\sqrt{i}}{2}$$

M: 2. rötter til $w = i = e^{i\frac{\pi}{2}}$



$$\underset{(w^{\frac{1}{2}})}{\tilde{z}_0} = e^{i\frac{\pi}{4}} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$$

$$\underset{(M)}{=} \frac{-1+i \pm \sqrt{2}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)}{2}$$

$$= \frac{-1+i \pm (1+i)}{2}$$

Så: $z_0 = \frac{-1+i+1+i}{2} = \underline{\underline{i}}$

$$z_1 = \frac{-1+i-1-i}{2} = \underline{\underline{-1}}$$