Plenum 10/9-14

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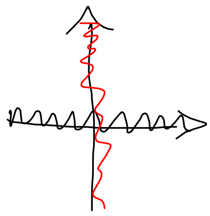
3.3: 6, 8, 9a
3.4: 1c, 3, 4, 9a, 11bc, 15
3.5: 1a, 3ab, 5, 7, 11, 13
4.3: 1d, 3c

3.3: Komplekse eksponensialer & De Moivre

6) $\cos(2\theta) + i\sin(2\theta) \underset{\text{De Moivre}}{=} (\cos\theta + i\sin\theta)^2$ $\boxed{\theta}$



$$= \cos^2\theta + 2i\cos\theta\sin\theta + i^2\sin^2\theta$$
$$= \cos^2\theta + 2i\cos\theta\sin\theta - \sin^2\theta$$
$$= (\cos^2\theta - \sin^2\theta) + i(2\cos\theta\sin\theta)$$

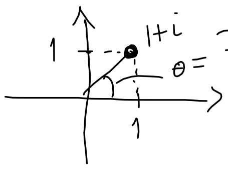
Imaginærdel pares med imaginærdel og realdel med realdel på venstre og høyre side:

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta, \quad \sin(2\theta) = 2\cos\theta\sin\theta$$
$$(\cos^2\theta + \sin^2\theta = 1 \Rightarrow \sin^2\theta = 1 - \cos^2\theta)$$

8.) $(1+i)^{804}$:

Skriver på polarform: $1+i$: $r = \sqrt{1^2+1^2} = \underline{\sqrt{2}}$

$\theta = \frac{\pi}{4}$;



$1 = \sqrt{2}\cos\theta$, $1 = \sqrt{2}\sin\theta$

Så: $1+i = \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)$

Derfor er:

$$(1+i)^{804} = \left(\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)\right)^{804}$$

$$= (\sqrt{2})^{804}\left(\cos\left(\frac{804\pi}{4}\right) + i\sin\left(\frac{804\pi}{4}\right)\right)$$

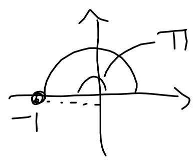
De moivres formel

$\sqrt{2} = 2^{\frac{1}{2}}$

$$= 2^{402}\left(\cos(201\pi) + i\sin(201\pi)\right)$$

$$= 2^{402}\left(\cos(100\cdot 2\pi + \pi) + i\sin(100\cdot 2\pi + \pi)\right)$$

$$= 2^{402}\left(\cos(\pi) + i\sin(\pi)\right)$$



$$= 2^{402}(-1 + i\cdot 0) = \underline{\underline{-2^{402}}}$$

3.4: Røtter av komplekse tall

1) c) $z = 2 + 2i\sqrt{3}$:

Eksponensialform: $r = |z| = \sqrt{2^2 + 2^2\sqrt{3}^2}$

$= \sqrt{4 + 4 \cdot 3} = \sqrt{16} = 4$

$$\begin{aligned} 2 &= 4\cos\theta \\ 2\sqrt{3} &= 4\sin\theta \end{aligned} \Rightarrow \begin{aligned} \cos\theta &= \frac{1}{2} \\ \sin\theta &= \frac{\sqrt{3}}{2} \end{aligned} \Rightarrow \theta = \frac{\pi}{3}$$

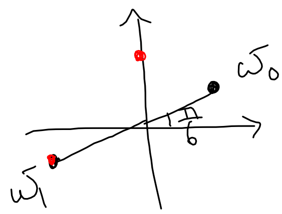
$z = 4e^{\frac{\pi}{3}i}$

Eksakte verdier:

	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
sin	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

Rot 1: $w_0 = z^{\frac{1}{2}} = \left(4e^{\frac{\pi}{3}i}\right)^{\frac{1}{2}}$

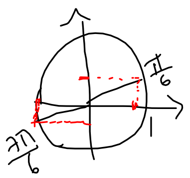
$= 2e^{i\frac{\pi}{6}} = 2\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)$



$= 2\left(\frac{\sqrt{3}}{2} + i\,\frac{1}{2}\right) = \sqrt{3} + i$

Rot 2: $w_1 = w_0\, e^{i\frac{2\pi}{2}} = 2e^{i\frac{\pi}{6}}e^{i\pi}$

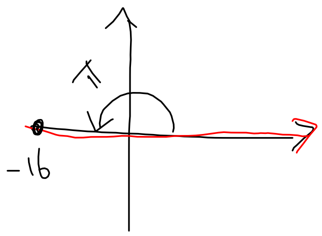
2. rotter



$= 2e^{i\frac{7\pi}{6}} = 2\left(\cos\left(\frac{7\pi}{6}\right) + i\sin\left(\frac{7\pi}{6}\right)\right)$

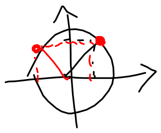
$= 2\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = -\sqrt{3} - i$

4)a) $\underline{z=-16}$: Eksponensialform: $z = 16e^{i\pi}$



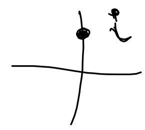
Rot 1: $w_0 = z^{\frac{1}{4}} = (16e^{i\pi})^{\frac{1}{4}}$

$= 16^{\frac{1}{4}} e^{i\frac{\pi}{4}} = 2e^{i\frac{\pi}{4}}$



$= 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = 2\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$

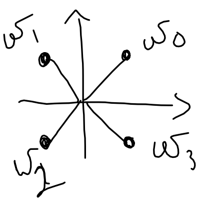
$= \sqrt{2} + \sqrt{2}i$



Må gange med $e^{i\frac{2\pi}{4}} = e^{i\frac{\pi}{2}} = i$ for å finne neste rot.

4. rotter

Rot 2: $w_1 = w_0 e^{i\frac{\pi}{2}} = 2e^{(\frac{\pi}{4}+\frac{\pi}{2})i} = 2e^{\frac{3\pi}{4}i}$



$= 2\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)$

$= 2\left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = -\sqrt{2} + \sqrt{2}i$

Rot 3: $w_2 = w_1 e^{i\frac{\pi}{2}} = 2\left(e^{i(\frac{3\pi}{4}+\frac{\pi}{2})}\right) = 2e^{\frac{5\pi}{4}i}$

$= -\sqrt{2}i - \sqrt{2} = -\sqrt{2} - \sqrt{2}i$

Rot 4: $w_3 = w_2 e^{i\frac{\pi}{2}} = 2e^{\frac{7\pi}{4}i}$

$= \sqrt{2} - \sqrt{2}i$

11) c) $z^2 + 2z - i\sqrt{3} = 0$

$$z = \frac{-2 \pm \sqrt{4 - 4\cdot 1\cdot(-i\sqrt{3})}}{2\cdot 1}$$

$$= \frac{-2 \pm \sqrt{4 + 4i\sqrt{3}}}{2} = \frac{-2 \pm \sqrt{4(1+i\sqrt{3})}}{2}$$

Mellom: $\underbrace{\sqrt{4+4i\sqrt{3}}}_{y}$: Polarform $y = 8e^{\frac{\pi}{3}i}$

Rot 1: $y^{\frac{1}{2}} = 8^{\frac{1}{2}} e^{\frac{\pi}{6}i} = 2\sqrt{2}\, e^{\frac{\pi}{6}i}$

$8^{\frac{1}{2}} = (4\cdot 2)^{\frac{1}{2}} = 4^{\frac{1}{2}} \cdot 2^{\frac{1}{2}}$

$$z = -1 \pm 2\sqrt{2}\, e^{\frac{\pi}{6}i} = -1 \pm \left(\frac{\sqrt{6}}{2} + \frac{\sqrt{2}}{2} i\right)$$

3.5: Algebraens fundamentalteorem

TRIKS: $w := z^2$

3) a) $z^4 + 2z^2 + 1 = 0$

$w^2 + 2w + 1 = 0$

$$w = \frac{-2 \pm \sqrt{4-4}}{2} = \frac{-2 \pm 0}{2} = -1$$

$\Rightarrow z^2 = w = -1 \Rightarrow z = \pm i$

Reelt polynom $\Rightarrow$ Konjugerte røtter har samme multiplisitet.

$\downarrow$

$\boxed{\text{Lemma } 3.5.4}$

Har to røtter $\pm i$ og 4. gradspolynom $\Rightarrow$

Begge har multiplisitet to

Kompleks faktorisering:

$$z^4 + 2z^2 + 1 = \underline{\underline{(z-i)^2 (z+i)^2}}$$

Reell faktorisering:

$$z^4 + 2z^2 + 1 = [(z-i)(z+i)]^2$$

$$\underset{\substack{\downarrow \\ \text{3. kvad. set.}}}{=} [z^2 - (-1)]^2 = \underline{\underline{(z^2+1)^2}}$$

b) $z^3 + 2z^2 + 4z = 0$

$$z(z^2 + 2z + 4) = 0$$

$$z = 0 \quad \text{eller} \quad \underbrace{z^2 + 2z + 4 = 0}_{\text{2. gradsformel}}$$

7.) a) $P(r) = P(1-2i) = (1-2i)^3 + 2(1-2i)^2$

$$- 3(1-2i) + 20$$

$$= -11 + 2i + 2(-3-4i) - 3 + 6i + 20$$

$$= \cancel{-11}\cancel{+20} - \cancel{3} - \cancel{6} + \cancel{2i} - \cancel{8}i + \cancel{6}i$$

$= 0$

så $P(r) = 0$, altså er r en rot i $P(z)$.

b) P er et reelt polynom $\Rightarrow$ $\bar{r} = 1 + 2i$ er også en rot.

P er delelig med $(z - r)(z - \bar{r}) = z^2 - 2z + 5$

Polynomdivisjon:

$$
\begin{aligned}
&z^3 + 2z^2 - 3z + 20 \div z^2 - 2z + 5 = \underline{z + 4} \\
&\underline{-(z^3 - 2z^2 + 5z)} \\
&\qquad 4z^2 - 8z + 20 \\
&\qquad \underline{-(4z^2 - 8z + 20)} \\
&\qquad\qquad 0
\end{aligned}
$$

Kompleks faktorisering:

$$P(z) = \underline{\underline{(z - r)(z - \bar{r})(z + 4)}}$$

Reelle faktorisering:

$$P(z) = \underline{\underline{(z^2 - 2z + 5)(z + 4)}}$$