Plenum 13/11-14

9.4: 8, 10, 11

9.5: 1ace, 3abde, 5, 13, 14

FVLA:

1.1: 1, 2, 3 c), 5

1.2: 1, 4, 6, 7, 13, 15, 17, 21, 25

9.4: Spesielle teknikker

11) $$\int \frac{\sin^3(2x)}{\sqrt{\sin x}}\,dx = \int \sin^3(2x)\,(\sin x)^{-\frac{1}{2}}\,dx$$

$$= \int (2\sin x\cos x)^3 (\sin x)^{-\frac{1}{2}}\,dx = 8\int \sin^3(x)\cos^{\overset{\text{odde!}}{3}}(x)\sin^{-\frac{1}{2}}(x)\,dx$$

$$= 8\int \sin^{\frac{5}{2}}(x)\cos^3(x)\,dx \underset{\text{TRIKS!}}{=} 8\int \sin^{\frac{5}{2}}(x)(1-\sin^2(x))\cos(x)\,dx$$

$$\boxed{\frac{5}{2}+\frac{4}{2}=\frac{9}{2}}$$

$$= 8\int \sin^{\frac{5}{2}}(x)\cos(x)\,dx - 8\int \sin^{\frac{9}{2}}(x)\cos(x)\,dx$$

$$= 8\int u^{\frac{5}{2}}\,du - 8\int u^{\frac{9}{2}}\,du = 8\,\frac{2}{7}\,u^{\frac{7}{2}} - 8\,\frac{2}{11}\,u^{\frac{11}{2}} + C$$

$u = \sin x$

$du = \cos x\,dx$

$\frac{du}{\cos x} = dx$

$$= \frac{16}{7}(\sin(x))^{\frac{7}{2}} - \frac{16}{11}(\sin(x))^{\frac{11}{2}} + C$$

9.5: Uegentlige integraler

1) c) $\int_1^2 \frac{1}{x-1}\,dx = \int_0^1 \frac{1}{u}\,du$ ⊳ Divergerer fra Setning 9.5.8.

$u = x-1$
$du = dx$
$x=1 \Rightarrow u=0$
$x=2 \Rightarrow u=1$

5) $\int_1^\infty \frac{\ln x}{x^p}\,dx$? Gjør generell grensesammenligning med $\frac{1}{x^k}$:

$$\lim_{x\to\infty} \frac{\frac{\ln x}{x^p}}{\frac{1}{x^k}} = \lim_{x\to\infty} \frac{\overset{\to\infty}{\ln x}}{x^{p-k}} = (\Delta)$$

$$\to \begin{cases} \infty & \text{hvis } p>k \\ 1 & \text{hvis } p=k \\ 0 & \text{hvis } p<k \end{cases}$$

Så hvis $p=k$ eller $p<k$ er grensen ∞, og dermed ikke konvergens (grensesml. testen).

Hvis $p>k$:

$$(\Delta) \underset{\text{"}\frac{\infty}{\infty}\text{" : L'H}}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{(p-k)x^{p-k-1}} = \lim_{x\to\infty} \frac{1}{(p-k)\underbrace{x^{p-k}}_{\to\infty \text{ siden } p>k}} = 0 < \infty$$

Så siden $\int_1^\infty \frac{1}{x^k}\,dx$ konv. for $k>1$, og p må være større enn k, vil $\int_1^\infty \frac{\ln x}{x^p}\,dx$ konv. for alle $p>1$.

13.) a) $I_n = \int_0^1 x(\ln x)^n \, dx$

VIS: For alle $n \in \mathbb{N}$ er $I_n = -\frac{n}{2} I_{n-1}$: (★)

Induksjonsbevis:

Viser først at (★) er OK for $n=1$:

$$I_1 = \int_0^1 x \ln x \, dx = \lim_{a \to 0} \left[\frac{1}{2}x^2 \ln x\right]_{x=a}^1 - \int_0^1 \frac{1}{2}x^2 \frac{1}{x} \, dx$$

$u = \ln x$
$v' = x$
$\Downarrow$
$v = \frac{1}{2}x^2$
$u' = \frac{1}{x}$

$$= -\frac{1}{2} \lim_{a \to 0} a^2 \ln a - \frac{1}{4}$$

M: $\lim_{a \to 0} a^2 \ln a = \lim_{a \to 0} \frac{\ln a}{\frac{1}{a^2}}$ ($a^2 \to 0$, $\ln a \to -\infty$)

$$= \lim_{a \to 0} \frac{\frac{1}{a}}{-2\frac{1}{a^3}} = \lim_{a \to 0} -\frac{1}{2}a^2 = \underline{0}$$

"$\frac{\infty}{\infty}$": L'H

$$I_1 = -\frac{1}{2} \cdot 0 - \frac{1}{4} = \underline{-\frac{1}{4}}$$

$$I_0 = \int_0^1 x \, dx = \underline{\frac{1}{2}}$$

$$-\frac{1}{2} I_0 = -\frac{1}{2}\frac{1}{2} = -\frac{1}{4} = I_1$$

OK for $n=1$.

Hypotese: Anta at (★) holder for alle n opp til og med $k \in \mathbb{N}$.

Vil vise at da er (*) også sann for $k+1$:

$$I_{k+1} = -\frac{k+1}{2} I_k$$

Induksjonssteg:

$$I_{k+1} = \int_0^1 x (\ln x)^{k+1} dx$$

$$= \lim_{a \to 0} \left[\frac{1}{2} x^2 (\ln x)^{k+1} \right]_{x=a}^{1} - \underbrace{\int_0^1 \frac{1}{\cancel{x}} (k+1)(\ln x)^k \frac{1}{2} x^{\cancel{2}} dx}$$

$u = (\ln x)^{k+1}$
$v' = x$
$\Rightarrow$ $v = \frac{1}{2}x^2$
$u' = (k+1)(\ln x)^k \frac{1}{x}$

$$= -\frac{1}{2}(k+1) I_k - \frac{1}{2} \lim_{a \to 0} a^2 (\ln a)^{k+1}$$

M: $\lim_{a \to 0} \underset{\substack{\downarrow \\ 0}}{a^2} \underset{\substack{\downarrow \\ \infty}}{(\ln a)^{k+1}} = \lim_{a \to 0} \frac{(\ln a)^{k+1}}{\frac{1}{a^2}} = \lim_{a \to 0} \frac{(k+1)(\ln a)^k \frac{1}{a}}{-2a^{-3}}$ "$\frac{\infty}{\infty}$": L'H

$$= -\frac{k+1}{2} \lim_{a \to 0} \frac{(\ln a)^k}{\frac{1}{a^2}}$$

Merk: Samme vil skje når vi fortsetter! Graden til $(\ln a)^{\cdots}$ vil reduseres med én per runde L'H, men vi vil beholde $\frac{1}{a^2}$ i nevneren. Får bare noen ekstra konstanter per runde.

en konstant "$\frac{\infty}{\infty}$": L'H

$$\lim_{a \to 0} a^2 (\ln a)^{k+1} = \dots = K \lim_{a \to 0} \frac{\ln a}{\frac{1}{a^2}} = K \lim_{a \to 0} \frac{\frac{1}{a}}{-2\frac{1}{a^3}}$$

$$= C \lim_{a \to 0} a^2 = \underline{0}$$

Så: $I_{k+1} = -\frac{k+1}{2} I_k - \frac{1}{2}\cdot 0 = -\frac{k+1}{2} I_k$

som var det vi ville vise. ∎

VIS:

b) $I_n = (-1)^n \frac{n!}{2^{n+1}}$ $(\square)$

Bevis: $n=1$: VS: $I_1 \overset{a)}{=} -\frac{1}{4}$

HS: $(-1)^1 \frac{1!}{2^{1+1}} = -\frac{1}{4}$

$n+1 = 1+1 = 2$

$2+1 = 3 \ldots$

Så VS = HS $\Rightarrow$ $(\square)$ er OK for $n=1$.

Hypotese: Anta at $(\square)$ holder for alle n opp til $k \in \mathbb{N}$.

Induksjonssteg: Vil vise at da holder $(\square)$ også for $k+1$:

$$I_{k+1} \underset{a)}{=} -\frac{k+1}{2} I_k \underset{\text{Hypotese}}{=} -\frac{k+1}{2}(-1)^k \frac{k!}{2^{k+1}}$$

$$= (-1)^{k+1} \frac{(k+1)!}{2^{(k+1)+1}}$$

$k! = 1\cdot 2 \cdot \ldots \cdot k$

$(k+1)k!$

$= 1\cdot 2\cdot \ldots \cdot k\cdot(k+1)$

$= (k+1)!$

som er det vi ville vise. ∎

FVLA: 1.1: Algebra for n-tupler

3)c) VIS: $(\vec{x}+\vec{y})\cdot(\vec{x}-\vec{y}) = \vec{x}\cdot\vec{x} - \vec{y}\cdot\vec{y}$:

$$(\vec{x}+\vec{y})\cdot\underbrace{(\vec{x}-\vec{y})}_{\vec{z}} = \vec{x}\cdot\vec{z} + \vec{y}\cdot\vec{z} = \vec{x}\cdot(\vec{x}-\vec{y}) + \vec{y}\cdot(\vec{x}-\vec{y})$$

$\vec{z} := \vec{x}-\vec{y}$

Set. 1.1.1e

$$= \vec{x}\cdot\vec{x} - \vec{x}\cdot\vec{y} + \vec{y}\cdot\vec{x} - \vec{y}\cdot\vec{y}$$

Set. 1.1.1e

$$= \vec{x}\cdot\vec{x} - \cancel{\vec{x}\cdot\vec{y}} + \cancel{\vec{x}\cdot\vec{y}} - \vec{y}\cdot\vec{y}$$

Set. 1.1.b

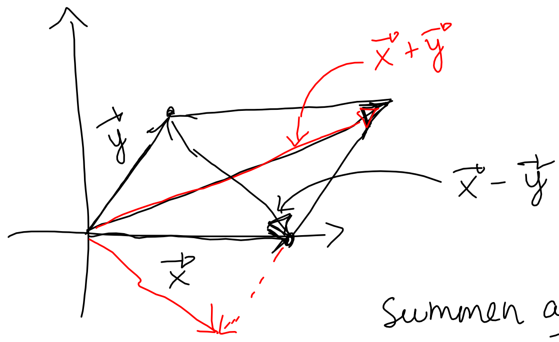
$$= \vec{x}\cdot\vec{x} - \vec{y}\cdot\vec{y}$$

$$\vec{x}\cdot(\vec{x}-\vec{y})$$
$$= \vec{x}\cdot(\vec{x}+\underbrace{(-\vec{y})}_{\vec{w}})$$
$$= \vec{x}\cdot(\vec{x}+\vec{w})$$
$$= \vec{x}\cdot\vec{x} + \vec{x}\cdot\vec{w} = \vec{x}\cdot\vec{x} + \vec{x}\cdot(-\vec{y})$$
$$= \vec{x}\cdot\vec{x} + (-\vec{x}\cdot\vec{y}) = \vec{x}\cdot\vec{x} - \vec{x}\cdot\vec{y}$$

1.2: Geometri for n-tupler

17.) VIS: $|\vec{x}+\vec{y}|^2 + |\vec{x}-\vec{y}|^2 = 2|\vec{x}|^2 + 2|\vec{y}|^2$

$$|\vec{x}+\vec{y}|^2 + |\vec{x}-\vec{y}|^2 = (\vec{x}+\vec{y})\cdot(\vec{x}+\vec{y}) + (\vec{x}-\vec{y})\cdot(\vec{x}-\vec{y})$$
$$= \vec{x}\cdot\vec{x} + \cancel{2\vec{x}\cdot\vec{y}} + \vec{y}\cdot\vec{y} + \vec{x}\cdot\vec{x} - \cancel{2\vec{x}\cdot\vec{y}} + \vec{y}\cdot\vec{y}$$
$$= 2\vec{x}\cdot\vec{x} + 2\vec{y}\cdot\vec{y} = 2|\vec{x}|^2 + 2|\vec{y}|^2$$



Summen av kvadratene av sidene:

$$|\vec{x}|^2 + |\vec{x}|^2 + |\vec{y}|^2 + |\vec{y}|^2$$
$$= 2|\vec{x}|^2 + 2|\vec{y}|^2$$

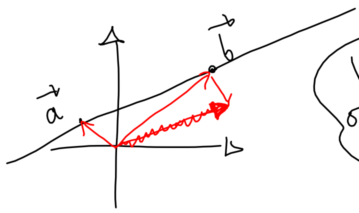
Sum kvadrat diagonalene:

$$|\vec{x}-\vec{y}|^2 + |\vec{x}+\vec{y}|^2$$

Disse er like! Så påstanden er OK.

21.) $\vec{a} = (7,-3,2,4,-2)$, $\vec{b} = (2,1,-1,-1,5)$

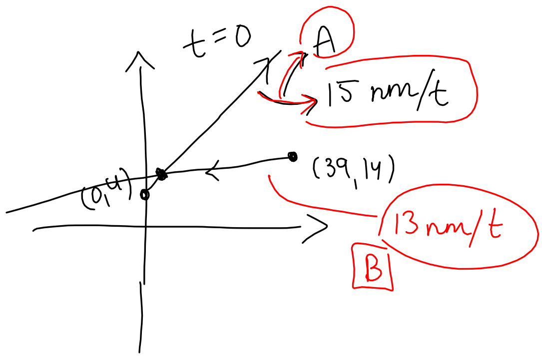
Retningsvektor: $\vec{b}-\vec{a} = (-5,4,-3,-5,7) := \vec{c}$



kunne også tatt $\vec{a}-\vec{b}$

$$\vec{r}(t) = \vec{a} + t\vec{c}$$
$$= (7-5t, -3+4t, 2-3t, 4-5t, -2+7t)$$

25.)



a) Parameterframstilliger:

A: $\vec{r}_A(t) = (0,4) + t(3,4)$
$$= (3t, 4t+4)$$

B: $\vec{r}_B(t) = (39, 14) + t(-12, 5) = (39 - 12t, 14 + 5t)$

Kryss: $3t_1 = 39 - 12t_2$, $4 + 4t_1 = 14 + 5t_2$

$\Downarrow$

$t_1 = 13 - 4t_2 \Rightarrow 4 + 4(13 - 4t_2) = 14 + 5t_2$

$\Downarrow$

$\underline{t_1 = 5} \Leftarrow \underline{t_2 = 2}$

Så parameterframstillingene krysser i $(3 \cdot 5, 4 + 4 \cdot 5) = \underline{\underline{(15, 24)}}$

(B: $(39 - 12 \cdot 2, 14 + 5 \cdot 2) = (39 - 24, 14 + 10) = (15, 24)$)

b) Kolliderer? Nei! Krysser kun én gang, i (15, 24).

A må flytte seg: $\sqrt{(15-0)^2 + (24-4)^2} = \underline{25 \text{ nm}}$

B må flytte seg: $\sqrt{(39-15)^2 + (14-24)^2} = \underline{26 \text{ nm}}$

A bruker: $\dfrac{25 \text{ nm}}{15 \frac{\text{nm}}{t}} = \dfrac{5}{3}$ timer

B bruker: $\dfrac{26 \text{ nm}}{13 \frac{\text{nm}}{t}} = 2$ timer

Så, siden tiden A bruker $\neq$ tiden B bruker, vil skipene ikke $\underline{\underline{\text{kollidere}}}$.