Plenum 19/11-14

1.3: 4

1.4: 4, 10

1.5: 11, 12

1.6: 5, 6, 12

1.3: Komplekse n-tupler

4) VIS: For alle ($\forall$) $\vec{x}, \vec{y} \in \mathbb{C}^n$ så er

$$|\vec{x}-\vec{y}|^2 = |\vec{x}|^2 - 2\mathrm{Re}(\vec{x}\cdot\vec{y}) + |\vec{y}|^2$$

Bevis:

$$|\vec{x}-\vec{y}|^2 = (\vec{x}-\vec{y})\cdot(\vec{x}-\vec{y}) = \vec{x}\cdot\vec{x} - \vec{x}\cdot\vec{y} - \vec{y}\cdot\vec{x} + \vec{y}\cdot\vec{y}$$

$$= |\vec{x}|^2 + |\vec{y}|^2 - \vec{x}\cdot\vec{y} - \overline{\vec{x}\cdot\vec{y}}$$

$$= |\vec{x}|^2 + |\vec{y}|^2 - 2\mathrm{Re}(\vec{x}\cdot\vec{y})$$

$\vec{x}\cdot\vec{y} \neq \vec{y}\cdot\vec{x}$ for $\vec{y}, \vec{x} \in \mathbb{C}^n$ generelt!

Generelt for $z \in \mathbb{C}$

$$z + \bar{z} = a + \cancel{ib} + a - \cancel{ib} = 2a = 2\mathrm{Re}\, z$$

VIS: $(\vec{x}+\vec{y})\cdot(\vec{x}-\vec{y}) = |\vec{x}|^2 - 2\mathrm{Im}(\vec{x}\cdot\vec{y})\, i - |\vec{y}|^2$

Bevis:

$$(\vec{x}+\vec{y})\cdot(\vec{x}-\vec{y}) = \vec{x}\cdot\vec{x} - \vec{x}\cdot\vec{y} + \vec{y}\cdot\vec{x} - \vec{y}\cdot\vec{y} = |\vec{x}|^2 - |\vec{y}|^2 - \vec{x}\cdot\vec{y} + \overline{\vec{x}\cdot\vec{y}}$$

$$= |\vec{x}|^2 + |\vec{y}|^2 - 2\,\mathrm{Im}(\vec{x}\cdot\vec{y})\,i$$

Generelt: $z \in \mathbb{C}$ så er

$$-z + \bar{z} = -\cancel{a} - ib + \cancel{a} - ib$$
$$= -2ib = 2\,\mathrm{Im}(z)\,i$$

1.4: Vektorproduktet

4.) $(2,0,-3)$ og $(-1,3,4)$:

$(2,0,-3)\times(-1,3,4) = \underline{\underline{(9,-5,6)}}$ står normalt på begge vektorene.

$$\begin{array}{cccccc}
\vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} & \vec{k} \\
2 & 0 & -3 & 2 & 0 & -3 \\
-1 & 3 & 4 & -1 & 3 & 4
\end{array}$$

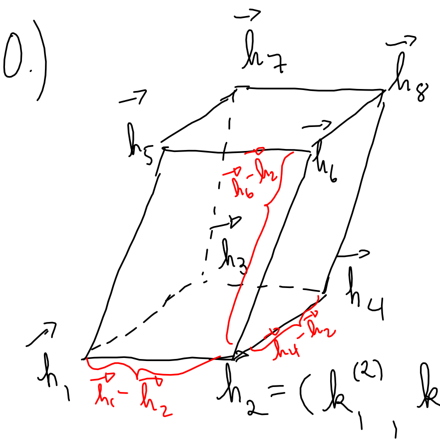
$$9\vec{i} - 8\vec{j} + 0\vec{k} + 0\vec{i} + 3\vec{j} + 6\vec{k}$$
$$= 9\vec{i} - 5\vec{j} + 6\vec{k} = \underline{(9,-5,6)}$$

Kan sjekke:

$(9,-5,6)\cdot(2,0,-3) = 18 - 18 = \underline{0}$

$(9,-5,6)\cdot(-1,3,4) = -9 - 15 + 24 = \underline{0}$

10.)



$\vec{h}_2 = (k_1^{(2)}, k_2^{(2)}, k_3^{(2)})$

Et parallellepiped har 8 hjørner. Kall disse $\vec{h}_1, \dots, \vec{h}_8$. Disse har heltallige koeff: $\vec{h}_i = (\underbrace{k_1^{(i)}}_{\in\mathbb{Z}}, \underbrace{k_2^{(i)}}_{\in\mathbb{Z}}, \underbrace{k_3^{(i)}}_{\in\mathbb{Z}})$

for $i = 1, \dots, 8$.

Parallellepipedet er utspent av $\vec{h}_4 - \vec{h}_2,\ \vec{h}_1 - \vec{h}_2,\ \vec{h}_6 - \vec{h}_2$

Volumet er da (Setning 1.4.4):

har heltallige koeff

$$\left| \underbrace{\underbrace{\underbrace{\left(\vec{h}_4 - \vec{h}_2 \right)}_{\text{har heltallige koeff.}} \times \underbrace{\left(\overset{\frown}{\vec{h}_1} - \vec{h}_2 \right)}_{\text{har heltall koeff.}}}_{\substack{\text{har heltallige koeff.} \\ \text{(fra def. av vektorproduktet)}}} \cdot \underbrace{\left(\vec{h}_6 - \vec{h}_2 \right)}_{\text{har heltallige koeff}}}_{\text{Et heltall (fra def. av prikkproduktet)}} \right|$$

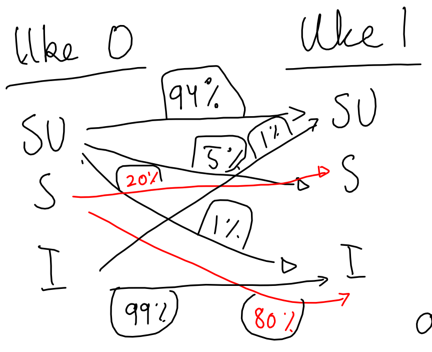
Bruker at heltall · heltall = heltall
heltall ± heltall = heltall

$\Downarrow$

Volumet er et heltall!

1.5: Matriser

11) 3 grupper: SU, S, I



$$\vec{v}_n = \begin{bmatrix} x_n \\ y_n \\ z_n \end{bmatrix} = \begin{bmatrix} \text{andel SU} \\ \text{—"— S} \\ \text{—"— I} \end{bmatrix}$$

a) Finn A s.a. $A\vec{v}_n = \vec{v}_{n+1}$

Se på piler inn!

$$x_{n+1} = 0{,}94\,x_n + 0\,y_n + 0{,}01\,z_n$$

$$y_{n+1} = 0{,}05\,x_n + 0{,}2\,y_n + 0\,z_n$$

$$z_{n+1} = 0{,}01\,x_n + 0{,}8\,y_n + 0{,}99\,z_n$$

$$A = \begin{bmatrix} 0{,}94 & 0 & 0{,}01 \\ 0{,}05 & 0{,}2 & 0 \\ 0{,}01 & 0{,}8 & 0{,}99 \end{bmatrix}$$

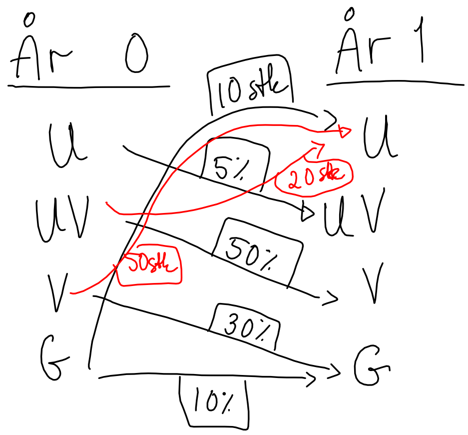
b) Uke 0: 10% S, 90% SU, 0 I $\Rightarrow \vec{v}_0 = \begin{bmatrix} 0{,}9 \\ 0{,}1 \\ 0 \end{bmatrix}$

Uke 1: $\vec{v}_1 = A\vec{v}_0 = \begin{bmatrix} 0{,}94 & 0 & 0{,}01 \\ 0{,}05 & 0{,}2 & 0 \\ 0{,}01 & 0{,}8 & 0{,}99 \end{bmatrix} \begin{bmatrix} 0{,}9 \\ 0{,}1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0{,}846 \\ 0{,}065 \\ 0{,}089 \end{bmatrix}$

$\Rightarrow$ 84,6% SU, 6.5% S, 8.9% I.

Uke 2: $\vec{v}_2 = A\vec{v}_1 \approx \begin{bmatrix} 0{,}796 \\ 0{,}055 \\ 0{,}149 \end{bmatrix} \Rightarrow$ 79,6% SU, 5.5% S, 14.9% I.

12.) Dyr: 4 grupper: U, UV, V, G



$$\vec{v}_n = \begin{bmatrix} x_n \\ y_n \\ z_n \\ u_n \end{bmatrix} = \begin{bmatrix} \text{\# U etter n år} \\ \text{\# UV — " —} \\ \text{\# V — " —} \\ \text{\# G — " —} \end{bmatrix}$$

a) Finn A s.a. $A\vec{v}_n = \vec{v}_{n+1}$:

$$A = \begin{array}{c|c} \begin{bmatrix} 0 & 20 & 50 & 10 \\ 0.05 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.3 & 0.1 \end{bmatrix} & \begin{matrix} U \\ UV \\ V \\ G \end{matrix} \\ \begin{matrix} U & UV & V & G \end{matrix} & \end{array}$$

b) År 0: 100 V $\Rightarrow \vec{v}_0 = \begin{bmatrix} 0 \\ 0 \\ 100 \\ 0 \end{bmatrix}$. $\rightarrow 4\times\cancel{4} \cdot \cancel{4}\times 1 = 4\times 1$

År 1: $\vec{v}_1 = A\vec{v}_0 = \begin{bmatrix} 0 & 20 & 50 & 10 \\ 0.05 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.3 & 0.1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 100 \\ 0 \end{bmatrix}$

$= \begin{bmatrix} 5000 \\ 0 \\ 0 \\ 30 \end{bmatrix} \rightarrow$ 5000 U, 0 UV, 0 V, 30 G.

<u>År 2</u>: $\vec{v}_2 = A\vec{v}_1 = \begin{bmatrix} 300 \\ 250 \\ 0 \\ 3 \end{bmatrix}$

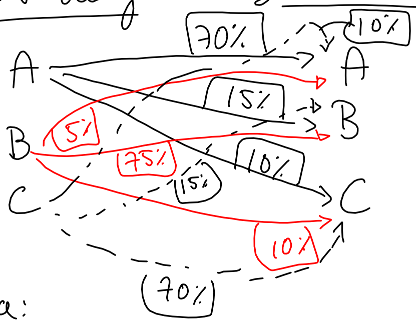
Etter 2 år er det 300 U, 250 UV, 0 V, 3 G.

$$\vec{v}_{n+1} = A\vec{v}_n = A(A\vec{v}_{n-1}) = (A\,A)\vec{v}_{n-1} = A^2\,\vec{v}_{n-1} = A^2(A\vec{v}_{n-2})$$
$$= A^3\vec{v}_{n-2} = \dots = \underline{\underline{A^{n+1}\vec{v}_0}}$$

1.6: Multiplikasjon av matriser

12) <u>Barnehage</u>: 3 sandkasser: A, B, C

Start dag Slutt dag



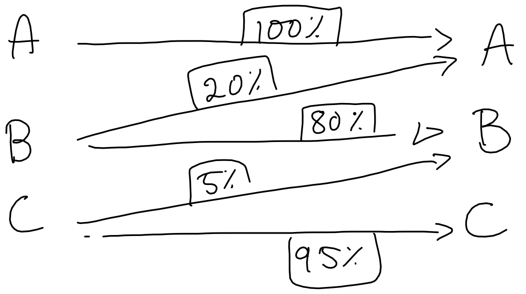
$\vec{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \text{\# l sand i A v/ start} \\ \text{— " — B — " —} \\ \text{— " — C — " —} \end{bmatrix}$

<u>Finn M s.a:</u>

a) $\vec{v} = M\vec{u}$ = sand i kassene ved slutten av dagen.

$$M = \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0.7 & 0.05 & 0.1 \\ 0.15 & 0.75 & 0.15 \\ 0.1 & 0.1 & 0.7 \end{bmatrix} \end{array} \begin{array}{c} A \\ B \\ C \end{array}$$

b) <u>Slutt dag</u> <u>Etter personale</u>



Finn N s.a. $\vec{w} = N\vec{v}$ gir hvor mye sand det er i hver kasse :

$$N = \begin{bmatrix} 1 & 0.2 & 0 \\ 0 & 0.8 & 0.05 \\ 0 & 0 & 0.95 \end{bmatrix}$$

c) $$K = NM = \begin{bmatrix} 1 & 0.2 & 0 \\ 0 & 0.8 & 0.05 \\ 0 & 0 & 0.95 \end{bmatrix} \begin{bmatrix} 0.7 & 0.05 & 0.1 \\ 0.15 & 0.75 & 0.15 \\ 0.1 & 0.1 & 0.7 \end{bmatrix}$$

$3\times\cancel{3}\cdot\cancel{3}\times 3$

$$= \begin{bmatrix} 0.73 & 0.2 & 0.13 \\ 0.125 & 0.605 & 0.155 \\ 0.095 & 0.095 & 0.665 \end{bmatrix}$$

$$\vec{u} = \begin{bmatrix} 200 \\ 300 \\ 400 \end{bmatrix}$$

Fordeling etter personale $= \vec{w} \overset{b)}{=} N\vec{v} \overset{a)}{=} N(M\vec{u})$

set. 1.6.2 i) $= (NM)\vec{u} = K\begin{bmatrix} 200 \\ 300 \\ 400 \end{bmatrix} = \begin{bmatrix} 258 \\ 268.5 \\ 313.5 \end{bmatrix}$

d) Neste dag:

$$\text{Fordeling etter personale dag 2} = \overbrace{(NM)}^{K}\vec{w} = \begin{bmatrix} 282,8 \\ 243,3 \\ 258,5 \end{bmatrix}$$

6.) $AB = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 0\cdot 1 + 1\cdot 3 & 0\cdot 1 + 4\cdot 1 \\ 0\cdot 1 + 2\cdot 3 & 0\cdot 1 + 2\cdot 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$

$AC = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$ Så $AB = AC$ selv $B \neq C$.

I motsetning til for reelle tall

$AD = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 3 & 7 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Så $AD = 0$ selv om $A \neq 0$ og $D \neq 0$.

nullmatrisen!