Plenum 22/10-14

8.2: 1, 5

8.3: 1, 3e, 5c, 6, 7a, 9

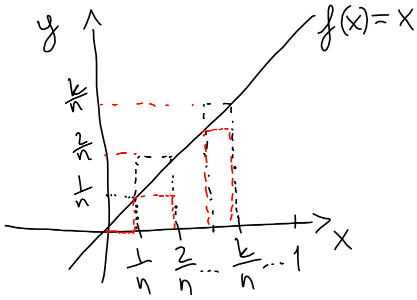
8.4: 1e, 3bcd, 5

8.2: Definisjon av integralet

5) $f:[0,1]\to\mathbb{R}$, $f(x)=x$, $\Pi_n=\{0,\frac{1}{n},\frac{2}{n},\dots,\frac{n-1}{n},1\}$

$\hookrightarrow \Delta x=\frac{1}{n}$

a)



$$\emptyset(\Pi_n)=\frac{1}{n}\sum_{k=1}^{n}\frac{k}{n}=\frac{1}{n^2}\sum_{k=1}^{n}k$$

$$\underset{\boxed{\text{Oppg. 1.2.1}}}{=}\frac{1}{n^2}\,\frac{n^2+n}{2}=\frac{1}{2}\left(1+\frac{1}{n}\right)$$

$$N(\Pi_n)=\frac{1}{n}\sum_{k=1}^{n}\frac{k-1}{n}=\frac{1}{n^2}\sum_{k=0}^{n-1}k=\frac{1}{n^2}\sum_{k=1}^{n-1}k$$

$y := k-1$

$k=1 \Rightarrow y=0$

$k=n \Rightarrow y=n-1$

$\to \sum_{y=0}^{n-1}\frac{y}{n}$

$$\underset{\boxed{\text{Oppg. 1.2.1}}}{=}\frac{1}{n^2}\left(1-\frac{1}{n}\right)$$

b) $\overline{\int_0^1} x\,dx = \inf\{\emptyset(\Pi) \mid \Pi \text{ er en partisjon av } [0,1]\}$

$= \lim_{n\to\infty} \emptyset(\Pi_n) \underset{a)}{=} \lim_{n\to\infty}\left(\frac{1}{2}\left(1+\frac{1}{n}\right)\right)$

Korollar 8.2.4: $f(x)=x$ er monoton $\Rightarrow$ holder å se på partisjoner med like store biter

$= \underline{\underline{\frac{1}{2}}}$

Tilsvarende:

$\underline{\int_0^1} x\,dx = \lim_{n\to\infty}\left(\frac{1}{2}\left(1-\frac{1}{n}\right)\right) = \underline{\underline{\frac{1}{2}}}$

c) Fra b) er f integrerbar siden

$\overline{\int_0^1} x\,dx = \underline{\int_0^1} x\,dx = \frac{1}{2} = \int_0^1 x\,dx$.

8.3: Analysens fundamentalteorem

3) e) $\int_0^1 \frac{1}{\sqrt{9-x^2}}\,dx = \int_0^1 \frac{1}{\sqrt{9\left(1-\frac{x^2}{9}\right)}}\,dx$

Framprovosere $\frac{1}{\sqrt{1-(\text{"noe"})^2}}$

NB: $D[\arcsin x] = \frac{1}{\sqrt{1-x^2}}$

$= \int_0^1 \frac{1}{3\sqrt{1-\left(\frac{x}{3}\right)^2}}\,dx = \frac{1}{3}\int_0^1 \frac{1}{\sqrt{1-\left(\frac{x}{3}\right)^2}}\,dx$

$= \frac{1}{\cancel{3}}\left[\cancel{3}\arcsin\left(\frac{x}{3}\right)\right]_{x=0}^{1} = \arcsin\left(\frac{1}{3}\right) - \arcsin(0)$

$= \underline{\underline{\arcsin\left(\frac{1}{3}\right)}}$

5) c) $f(x) = \int_1^x \arctan(t^2)\,dt$

$f'(x) = \underline{\underline{\arctan(x^2)}}$ → Fra Analysens fundamental-teorem

6) Vis: $G(x) := \int_a^{g(x)} f(t)\,dt, \quad a \in \mathbb{R}$

$G'(x) = f(g(x))\,g'(x).$

$D\Big[\underbrace{\int_0^{\arctan(x)} 2t\,dt}_{\text{funksjon av } x \,!!}\Big]$

$\int_x^0 \ldots dt = -\int_0^x \ldots dt$

Bevis: Definer $F(y) := \int_a^{y} f(t)\,dt$

Fra Analysens fundamentalteorem er
$F'(y) = f(y).$

Per definisjon er $G(x) = F(g(x))$, så fra kjerneregelen er

$G'(x) = F'(g(x))\,g'(x) = f(g(x))\,g'(x)$

som var det vi skulle vise. ⊠

b) i) $D\Big[\int_0^{\overbrace{\sin x}^{g(x)}} \underbrace{t e^{-t}}_{f(t)}\,dt\Big] \underset{\boxed{a)}}{=} \underline{\underline{\sin(x)\,e^{-\sin(x)}\cos(x)}}$

7.) a) $$\lim_{x\to 0} \frac{\int_0^x e^{-t^2}\,dt}{x} = \lim_{x\to 0} \frac{D[\int_0^x e^{-t^2}\,dt]}{1}$$

"$\frac{0}{0}$": L'H

$$= \lim_{x\to 0} e^{-x^2} = \underline{\underline{1}}$$

Analysens fundamentalteorem

Middelverdisetningen for integraler

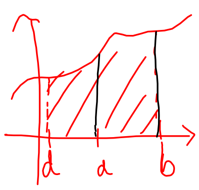
9.) Vis: Det fins $c \in (a,b)$ s.a. $\int_a^b f(x)\,dx = f(c)(b-a)$.

Bevis: La $F(x) := \int_d^x f(t)\,dt$, der $d \leq a$. Merk at F er veldef. siden f er kont., altså integrerbar.

Da er F kont. og deriverbar, og fra Analysens fundamentalteorem er $F'(x) = f(x)$. Se på $[a,b]$.

Da gir middelverdisetningen at det fins et tall $c \in (a,b)$ s.a.

$\int_d^b f(t)dt + \int_d^a f(t)dt - \int_d^a f(t)dt$



$$\underbrace{F'(c)} = \frac{F(b)-F(a)}{b-a}$$

$$\underline{f(c)(b-a)} = F(b)-F(a) = \int_d^b f(t)dt - \int_d^a f(t)dt$$
$$= \underline{\int_a^b f(t)\,dt}.$$

8.4: Det ubestemte integralet

1)e) $$\int \frac{4}{\sqrt{7-x^2}}\,dx = 4\int \frac{1}{\sqrt{7\left(1-\frac{x^2}{7}\right)}}\,dx$$

$D[\arcsin y] = \frac{1}{\sqrt{1-y^2}}$

$$= 4\int \frac{1}{\sqrt{7}\sqrt{1-\left(\frac{x}{\sqrt{7}}\right)^2}}\,dx = \frac{4}{\sqrt{7}}\int \frac{1}{\sqrt{1-\left(\frac{x}{\sqrt{7}}\right)^2}}\,dx$$

$$= \frac{4}{\cancel{\sqrt{7}}}\arcsin\left(\frac{x}{\sqrt{7}}\right)\cancel{\sqrt{7}} + C = 4\arcsin\left(\frac{x}{\sqrt{7}}\right)+C$$

3) b) $$\int \sin(2x)\,\frac{e^{\cos^2 x}}{e^{\sin^2 x}}\,dx = \int \sin(2x)\,e^{\cos^2 x - \sin^2 x}\,dx$$

$$= \int \sin(2x)\,e^{\cos(2x)}\,dx = -\frac{1}{2}e^{\cos(2x)} + C$$

c) $$\int \frac{1}{\sqrt{x}\,(1+x)}\,dx = \int \frac{1}{\sqrt{x}\,(1+(\sqrt{x})^2)}\,dx$$

$$= 2\int \frac{1}{2\sqrt{x}\,(1+(\sqrt{x})^2)}\,dx = 2\arctan(\sqrt{x}) + C$$

$\sqrt{x} = x^{\frac{1}{2}}$

$D[\sqrt{x}] = D[x^{\frac{1}{2}}] = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2}\frac{1}{\sqrt{x}} = \frac{1}{2\sqrt{x}}$

d) $$\int \frac{7x-1}{\sqrt{1-x^2}}\,dx = \int \frac{7x}{\sqrt{1-x^2}}\,dx - \int \frac{1}{\sqrt{1-x^2}}\,dx$$

$$= 7 \int \frac{x}{\sqrt{1-x^2}}\,dx - \arcsin(x) + C$$

$$\overset{\text{I}}{=} -7\sqrt{1-x^2} - \underline{\underline{\arcsin(x) + C}}$$

$$D\left[(1-x^2)^{\frac{1}{2}}\right] = \frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x) = -\frac{x}{\sqrt{1-x^2}}$$

5.) $f : (0,\infty) \to \mathbb{R}$, $f(xy) = f(x) + f(y)$ $(\star)$

for alle $x, y \in (0,\infty)$.

$(\star\star)$ f er deriverbar i $x=1$ med $f'(1) = k$

a) Vis $f(1)=0$: Velg $x = y = 1$

$$f(1) = f(1\cdot 1) = f(x\cdot y) \overset{(\star)}{=} f(x) + f(y) = f(1) + f(1) = 2f(1)$$

Så: $f(1) = 2f(1)$

$$\underline{\underline{0 = f(1)}}$$

b) Vis: $f(x+h) = f(x) + f\left(1+\frac{h}{x}\right)$:

$$f(x) + f\left(1+\frac{h}{x}\right) \overset{(\star)}{=} f\left(x\left(1+\frac{h}{x}\right)\right) = f(x+h)$$

Bruk dette og vis $f'(x) = \frac{k}{x}$:

$$f'(x) \underset{\text{Def. av derivert}}{=} \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \underset{\boxed{\text{over}}}{=} \lim_{h \to 0} \frac{\cancel{f(x)} + f(1+\frac{h}{x}) - \cancel{f(x)}}{h}$$

$$= \lim_{h \to 0} \frac{f(1+\frac{h}{x})}{h} = \lim_{h \to 0} \frac{f(1+\frac{h}{x})}{x \frac{h}{x}}$$

$$= \frac{1}{x} \lim_{h \to 0} \frac{f(1+\frac{h}{x})}{\frac{h}{x}} = \frac{1}{x} \lim_{y \to 0} \frac{f(1+y)}{y}$$

x er konstant i denne grense ⇒ OK å flytte ut!

La $y := \frac{h}{x}$

$h \to 0 \Leftrightarrow y \to 0$

$$= \frac{1}{x} \underbrace{\lim_{y \to 0} \frac{f(1+y) - f(1)}{y}}_{\text{Def. av } f'(1)} = \frac{1}{x} f'(1) = \frac{k}{x}$$

a): $f(1) = 0$

c) $f'(x) = \frac{k}{x} \Rightarrow f(x) = k \ln(x) + C$

Analysens fundamentalteorem

Men: $f(1) = k \ln(1) + C = 0 + C = C$

(a) $f(1) = 0$

$C = 0$, så $f(x) = k \ln(x)$.