Plenum 29/10-14

8.5: 1a, 4, 5

8.6: 1, 3, 5 acd, 7 aceg, 9, 11ac, 15

9.1: 1, 3, 5, 9, 15, 19

8.5: Riemannsummer

4) VIS: $\lim_{n\to\infty} \frac{1}{n^{\frac{3}{2}}} \left(\sum_{i=1}^{n} \sqrt{i} \right) = \frac{2}{3}$

Se på: $\int_0^1 \sqrt{x}\, dx$. Vil vise at uttrykket over er en Riemannsum for denne.

La $\Pi_n = \{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, 1\}$ være en partisjon av $[0,1]$ og la $f(x) = \sqrt{x}$. La $U_n = \{\frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, 1\}$.

$$R_f(\Pi_n, U_n) = \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}) = \sum_{i=1}^{n} \sqrt{c_i} \left(\frac{i}{n} - \frac{i-1}{n}\right)$$

$$= \sum_{i=1}^{n} \sqrt{\frac{i}{n}} \, \frac{1}{n} = \frac{1}{n^{\frac{3}{2}}} \sum_{i=1}^{n} \sqrt{i}$$

Så $\frac{1}{n^{\frac{3}{2}}} \sum_{i=1}^{n} \sqrt{i}$ er er en Riemannsum for $\int_0^1 \sqrt{x}\, dx$.

Merk at når $n \to \infty$; $|\Pi_n| \to 0$.

Fra Kor. 8.5.4 er:

$$\lim_{n\to\infty} \frac{1}{n^{\frac{3}{2}}} \sum_{i=1}^{n} \sqrt{i} = \int_0^1 \sqrt{x}\,dx = \int_0^1 x^{\frac{1}{2}}\,dx$$

$$= \left[\frac{2}{3} x^{\frac{3}{2}}\right]_{x=0}^{1} = \frac{2}{3}(1-0) = \underline{\underline{\frac{2}{3}}}$$

5) $\lim_{n\to\infty} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{1}{\sqrt{i}}$:

$$\begin{cases} \Pi_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, 1\right\} \\ f(x) = \frac{1}{\sqrt{x}} \\ U_n = \left\{\frac{1}{n}, \frac{2}{n}, \dots, 1\right\} \end{cases}$$

$$R_f(\Pi_n, U_n) = \sum_{i=1}^{n} f\left(\frac{i}{n}\right) \frac{1}{n} = \sum_{i=1}^{n} \frac{\sqrt{n}}{\sqrt{i}} \frac{1}{n} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{1}{\sqrt{i}}$$

Fra Kor. 8.5.4 er (siden $|\Pi_n| = \frac{1}{n} \xrightarrow[n\to\infty]{} 0$):

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{1}{\sqrt{i}} = \int_0^1 \frac{1}{\sqrt{x}}\,dx = \int_0^1 x^{-\frac{1}{2}}\,dx$$

$$= \left[2x^{\frac{1}{2}}\right]_{x=0}^{1} = 2(1-0) = \underline{\underline{2}}$$

8.6: Anvendelser av integralet

7) e) $y = \sin(x^2)$, $x = 0$, $x = \sqrt{\pi}$:

$$V = \int_0^{\sqrt{\pi}} 2\pi\, x \sin(x^2)\,dx = \pi \int_0^{\sqrt{\pi}} 2x \sin(x^2)\,dx$$

$$= \pi\left[-\cos(x^2)\right]_{x=0}^{\sqrt{\pi}} = \pi(\cos 0 - \cos\pi)$$

$$= \pi(1 - (-1)) = \underline{\underline{2\pi}}$$

11) c) $\overset{f(x)}{y} = \frac{x^2}{2} - \frac{1}{4}\ln(x)$, $x = 1$, $x = e$:

$$L = \int_1^e \sqrt{1 + f'(x)^2}\,dx = \int_1^e \sqrt{1 + \left(x - \frac{1}{4x}\right)^2}\,dx$$

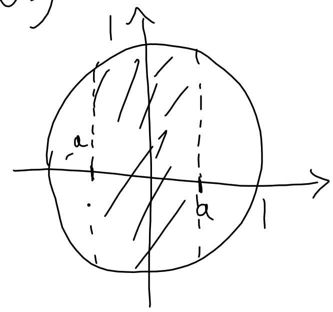
$$= \int_1^e \sqrt{1 + x^2 - \frac{2x}{4x} + \frac{1}{16x^2}}\,dx = \int_1^e \sqrt{\frac{1}{2} + x^2 + \frac{1}{16x^2}}\,dx$$

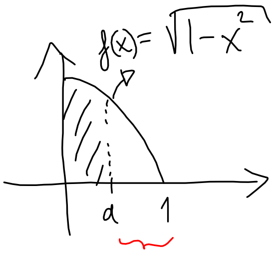
$$= \int_1^e \sqrt{\left(x + \frac{1}{4x}\right)^2}\,dx = \int_1^e \left(x + \frac{1}{4x}\right)dx = \left[\frac{1}{2}x^2 + \frac{1}{4}\ln(4x)\right]_{x=1}^e$$

$$= \frac{1}{2}e^2 + \frac{1}{4}(\cancel{\ln(4)} + \ln(e)) - \frac{1}{2} - \cancel{\frac{1}{4}\ln(4)}$$

$$= \underline{\underline{\frac{1}{2}e^2 - \frac{1}{4}}}$$

15.)





Kule m/ radius 1 og sentrum origo:

$$x^2 + y^2 = 1$$

$$y = \pm\sqrt{1-x^2}$$

$$V_{\text{omdreiningslegeme}} = 2\int_a^1 2\pi x\sqrt{1-x^2}\,dx$$

$$= 2\pi \int_a^1 2x \underbrace{\sqrt{1-x^2}}_{(1-x^2)^{\frac{1}{2}}}\,dx = 2\pi\left[-\frac{2}{3}(1-x^2)^{\frac{3}{2}}\right]_{x=a}^{1}$$

$$= \frac{4\pi}{3}(1-a^2)^{\frac{3}{2}}$$

9.1: Delvis integrasjon

9) $\int \sin(\ln x)\,dx = \int 1 \cdot \sin(\ln x)\,dx$

$$= x\sin(\ln x) - \int \cancel{x}\cos(\ln x)\frac{1}{\cancel{x}}\,dx + C$$

$u = \sin(\ln x)$
$v' = 1$
$u' = \cos(\ln x)\cdot\frac{1}{x}$
$v = x$

$$= x\sin(\ln x) - \int \cos(\ln x)\,dx + C$$

M: $\int \cos(\ln x)\,dx$

$= \int 1 \cdot \cos(\ln x)\,dx$

$u = \cos(\ln x)$
$v' = 1$
$u' = -\sin(\ln x)\frac{1}{x}$
$v = x$

$$= x\cos(\ln x) - \int \cancel{x}\left(-\sin(\ln x)\frac{1}{\cancel{x}}\right)dx$$

$$= x\cos(\ln x) + \int \sin(\ln x)\,dx$$

$$\int \sin(\ln x)\,dx = x\sin(\ln x) - x\cos(\ln x) - \int \sin(\ln x)\,dx + C$$

$$\Downarrow$$

$$2\int \sin(\ln x)\,dx = x\sin(\ln x) - x\cos(\ln x) + C$$

$$\int \sin(\ln x)\,dx = \frac{1}{2}x\left(\sin(\ln x) - \cos(\ln x)\right) + C$$

15) $y = \ln x$, $x \in [1,2]$ dreies om x-aksen:

$$V = \int_1^2 \pi\,(y(x))^2\,dx = \int_1^2 \pi \ln^2(x)\,dx$$

$$= \pi\int_1^2 1\cdot \ln^2(x)\,dx = \pi\left(\left[\ln^2(x)\,x\right]_{x=1}^{2} - \int_1^2 \cancel{x}\,2\ln(x)\frac{1}{\cancel{x}}\,dx\right)$$

$u = \ln^2(x)$

$v' = 1$

$\Downarrow$

$u' = 2\ln(x)\frac{1}{x}$

$v = x$

$$= \pi\left(2\ln^2(2) - 1\underbrace{\underbrace{\ln^2(1)}_{0}}_{0} - 2\int_1^2 \underbrace{\ln(x)}_{1\cdot\ln(x)}dx\right)$$

$$= 2\pi\left(\ln^2(2) - \left([x\ln x]_{x=1}^{2} - \int_1^2 \cancel{x}\frac{1}{\cancel{x}}dx\right)\right)$$

$u = \ln x$
$v' = 1$
$\Downarrow$
$u' = \frac{1}{x}$
$v = x$

$$= 2\pi\left(\ln^2(2) - 2\ln(2) + 0 + 2 - 1\right)$$

$$= 2\pi\left(\ln^2(2) - 2\ln(2) + 1\right)$$

19) $I_n = \int (\ln x)^n dx$

VIS: $I_n = x(\ln x)^n - nI_{n-1} \quad (*)$

Bevis: Induksjon: Vet at dette er OK for $n = 1$:

$$I_1 = \int \ln(x)\,dx = x\ln x - x + C$$

$\boxed{\text{EKS. 9.1.3}}$

$$= x(\ln x)^1 - 1 I_0$$

$$I_0 = \int (\ln x)^0 dx = \int 1\,dx = x + C$$

Hypotese: Anta at $(*)$ holder for alle $n \in \mathbb{N}$ opp til k.

Vil vise at (★) også holder for k+1.

$$I_{k+1} \overset{\text{Def. } I_k}{=} \int \underbrace{(\ln x)^{k+1}}_{1\cdot(\ln x)^{k+1}} dx = x(\ln x)^{k+1} - \int \cancel{x}(k+1)(\ln x)^k \frac{1}{\cancel{x}} dx$$

$u = (\ln x)^{k+1}$

$v' = 1$

$u' = (k+1)(\ln x)^k \frac{1}{x}$

$v = x$

$$= x(\ln x)^{k+1} - (k+1)\int (\ln x)^k dx$$

$$= x(\ln x)^{k+1} - (k+1) I_k$$

Def I_k

Dermed holder formelen. ⊠

Vet: $I_1 = x\ln x - x + C$

$$I_2 = x(\ln x)^2 - 2I_1 = x(\ln x)^2 - 2(x\ln x - x) + C$$

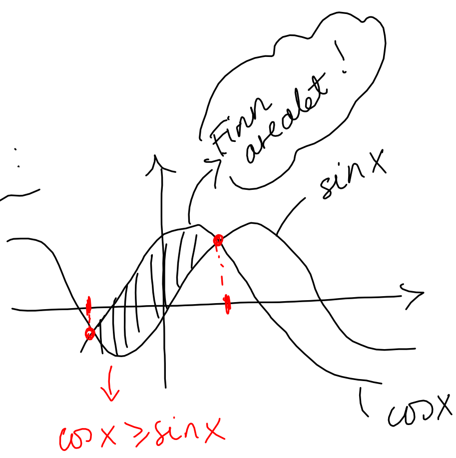
$$= x(\ln x)^2 - 2x\ln x + 2x + C$$

formel vist

$$I_3 = x(\ln x)^3 - 3I_2 = x(\ln x)^3 - 3x(\ln x)^2 + 6x\ln x - 6x + C$$

8.6:

3.)



<u>Skjæringspunkter:</u> $\sin x = \cos x \iff x = \frac{\pi}{4} + k\pi, \; k \in \mathbb{Z}.$

Fra figuren er "våre" skjæringspunkter $x = \frac{\pi}{4}$ og

$$x = \frac{\pi}{4} - \pi = -\frac{3\pi}{4}.$$

$$A = \int_{-\frac{3\pi}{4}}^{\frac{\pi}{4}} \cos x \, dx - \int_{-\frac{3\pi}{4}}^{\frac{\pi}{4}} \sin x \, dx$$

$$= \Big[\sin x\Big]_{x=-\frac{3\pi}{4}}^{\frac{\pi}{4}} + \Big[\cos x\Big]_{x=-\frac{3\pi}{4}}^{\frac{\pi}{4}}$$

$$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \frac{4\sqrt{2}}{2} = \underline{\underline{2\sqrt{2}}}$$

<u>9.1:</u>

5) $$\int \frac{\ln(x^2)}{x^2} dx = -\frac{\ln(x^2)}{x} - \int \frac{2}{x}\left(-\frac{1}{x}\right) dx$$

$u = \ln(x^2)$

$v' = \frac{1}{x^2} = x^{-2}$

$\Downarrow$

$u' = \frac{1}{\cancel{x^2}} 2\cancel{x} = \frac{2}{x}$

$v = -x^{-1} = -\frac{1}{x}$

$$= -\frac{\ln(x^2)}{x} + 2\int x^{-2} dx$$

$$= -\frac{\ln(x^2)}{x} + 2\left(-x^{-1}\right) + C$$

$$= -\frac{\ln(x^2)}{x} - \frac{2}{x} + C$$

$$= -\frac{2}{x}\left(\underline{\underline{\ln(x)+1}}\right) + C$$

$$\boxed{\ln(x^2) = \ln(x) + \ln(x) = 2\ln(x)}$$