Plenum 5/11-14

9.2: 1 abcg, 3 cd, 9, 11, 23, 29

9.3: 1d, 3ab, 5afg, 17, 21, 23, 25, 31

9.2: Substitusjon

3) c) $\displaystyle\int_4^9 \frac{\sqrt{x}+1}{1-\sqrt{x}}\,dx = \int_2^3 \frac{u+1}{1-u}\,2u\,du$

$u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}}\,dx$

$2u\,du = dx$

$x = 4 \Rightarrow u = 2$

$x = 9 \Rightarrow u = 3$

$\boxed{D(\sqrt{x}) = D(x^{\frac{1}{2}})}$

$= -2\displaystyle\int_2^3 \frac{u^2+u}{u-1}\,du = I$

M: Polynomdiv.

$u^2 + u \div u - 1 = u + 2 + \frac{2}{u-1}$

$-(u^2 - u)$

$2u$

$-(2u - 2)$

2

$I = -2\displaystyle\int_2^3 \left(u + 2 + \frac{2}{u-1}\right)du = -2\left[\frac{1}{2}u^2 + 2u + 2\ln|u-1|\right]_{u=2}^3$

$= -2\left(\frac{1}{2}9 + \cancel{6} + 2\ln 2 - \cancel{\frac{1}{2}4} - \cancel{4} - \underbrace{2\ln 1}_{0}\right)$

$= \underline{\underline{-4\ln 2 - 9}}$

d) $\int_0^3 \arctan(\sqrt{x})\,dx = \int_0^{\sqrt{3}} \arctan(u)\, 2u\, du$

$u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}}\,dx$

$2u\,du = dx$

$x = 0 \Rightarrow u = 0$

$x = 3 \Rightarrow u = \sqrt{3}$

$= 2\int_0^{\sqrt{3}} u \arctan(u)\,du = 2\left(\left[\frac{1}{2}u^2 \arctan u\right]_{u=0}^{\sqrt{3}} - \int_0^{\sqrt{3}} \frac{\frac{1}{2}u^2}{1+u^2}\,du \right)$

$v = \arctan u$

$w' = u$

$\Downarrow$

$v' = \frac{1}{1+u^2}$

$w = \frac{1}{2}u^2$

$= 3\arctan\sqrt{3} - \int_0^{\sqrt{3}} \frac{1+u^2-1}{1+u^2}\,du$

$= 3\arctan\sqrt{3} - \int_0^{\sqrt{3}} 1\,du + \int_0^{\sqrt{3}} \frac{1}{1+u^2}\,du$

$= 3\arctan\sqrt{3} - \sqrt{3} + \left[\arctan u\right]_{u=0}^{\sqrt{3}}$

$= 3\arctan\sqrt{3} - \sqrt{3} + \arctan\sqrt{3}$

$= 4\arctan\sqrt{3} - \sqrt{3}$

$= \frac{4\pi}{3} - \sqrt{3}$

29.) a) $L:(0,\infty)\to\mathbb{R}$, $L(x)=\int_1^x \frac{1}{t}\,dt$.

Hvorfor er:

- L veldef.? $\frac{1}{t}$ er kont. for alle $t\in[1,x]$ når $x\in(0,\infty)$, så dermed er den integrerbar (Analysens fundamentalteorem).

  Evt. $\frac{1}{t}$ er monoton i det aktuelle området

- L kont.? $\frac{1}{t}$ er kont. for alle aktuelle t, så fra def. av integralet er L kontinuerlig.

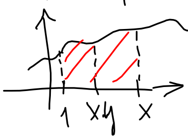
- L str. voksende? $\frac{1}{t}$ er str. pos. for alle $t\in(1,x)$, $x\in(0,\infty)$ $\Rightarrow$ integralet er str. voksende $\Rightarrow$ L str. voksende.

b) $$L(x\cdot y)=\int_1^{x\cdot y}\frac{1}{t}\,dt=\int_1^{x}\frac{1}{t}\,dt+\int_x^{xy}\frac{1}{t}\,dt$$

Def. L

Klart OK når $x\le xy$.

Hvis $xy<x$: $\int_1^{x}\frac{1}{t}\,dt+\int_x^{xy}\frac{1}{t}\,dt=\int_1^{x}\frac{1}{t}\,dt-\int_{xy}^{x}\frac{1}{t}\,dt$



$$= L(x) + \int_1^y \frac{1}{u\cancel{x}} \cancel{x}\, du = L(x) + \int_1^y \frac{1}{u}\, du$$

$$= L(x) + L(y)$$

Vil.:

$t = x \Rightarrow u = 1$

$u = \frac{t}{x}$ (OK: $x > 0$)

$t = xy \Rightarrow u = y$

$du = \frac{1}{x}\, dt$

$t = ux$

$$\boxed{L(y) = \int_1^y \frac{1}{t}\, dt}$$

c) $L(x^\alpha) = \alpha L(x)$:

• $\alpha \in \mathbb{N}$:

$$L(x^\alpha) = L(\underbrace{x \cdot x \cdot \ldots \cdot x}_{\alpha \text{ ganger}}) \overset{b)}{=} \underbrace{L(x) + L(x) + \ldots + L(x)}_{\alpha \text{ ganger}}$$

$$= \alpha L(x)$$

• $\alpha \in \mathbb{Z}$: La $m \in \mathbb{N}$, da er

$$0 \overset{\text{Def.}}{=} L(1) = L(x^m x^{-m}) \overset{b)}{=} L(x^m) + L(x^{-m})$$

$$\Downarrow$$

$$L(x^{-m}) = -L(x^m) \overset{\text{punkt over}}{=} -m L(x)$$

Så OK for $m \in \mathbb{Z}$.

• La $\alpha \in \mathbb{Z}, \alpha \neq 0$: $L(x^{\frac{1}{\alpha}}) = ?$

$$L(x) = L((\underbrace{x^{\frac{1}{\alpha}}}_{y})^\alpha) = L(y^\alpha) \overset{b)}{=} \alpha L(y)$$

$$= \alpha L(x^{\frac{1}{\alpha}})$$

Så: $L(x^{\frac{1}{\alpha}}) = \frac{1}{\alpha} L(x)$

• $\alpha = \frac{m}{n} \in \mathbb{Q}$:

$$L(x^{\frac{m}{n}}) = L((x^{\frac{1}{n}})^m) \underset{\text{tidligere pkt.}}{=} m\,L(x^{\frac{1}{n}}) = m\left(\frac{1}{n}L(x)\right)$$

$$= \frac{m}{n} L(x)$$

• $\alpha \in \mathbb{R}$: $L(x^{\alpha}) = L(\lim_{n \to \infty} x^{y_n}) = \lim_{n \to \infty} L(x^{y_n})$

$\{y_n\}$ er en følge i $\mathbb{Q}$ som konvergerer mot α

L kont. fra a)

over

$$= \lim_{n \to \infty} y_n L(x) = L(x) \lim_{n \to \infty} y_n = L(x)\alpha$$

$$= \alpha L(x)$$

d) $\lim_{x \to \infty} L(x) = \lim_{x \to \infty} L(2^x) = \lim_{x \to \infty} x\,L(2)$

$x \to \infty \Rightarrow 2^x \to \infty$

c)

$$= L(2) \lim_{x \to \infty} x \;\text{"="}\; L(2) \cdot \infty = \infty$$

$$\lim_{x \to 0^+} L(x) = \lim_{y \to \infty} L\left(\frac{1}{y}\right) = \lim_{y \to \infty} L(y^{-1})$$

$$= \lim_{y \to \infty} -L(y) = -\lim_{y \to \infty} L(y) = -\infty$$

9.3: Delbrøksoppspaltning

25.) a) $(2+i)^3 - 11(2+i) + 20 = (4+4i-1)(2+i)$

$-22 - 11i + 20 = \cancel{8} + 4\cancel{i} + \cancel{8}i - \cancel{4} - \cancel{2} - \cancel{i} - \cancel{2} - \cancel{11}i$

$= 0$

$\Rightarrow 2+i$ er en rot i $z^3 - 11z + 20$.

Reelt polynom $\Rightarrow$ Røttene kommer i kompleks konjugerte par

$\Rightarrow 2-i$ er en rot i polynomet.

$$z^3 - 11z + 20 = \underbrace{(z-(2+i))(z-(2-i))}(z - \ldots)$$

$$(z-(2+i))(z-(2-i)) = z^2 - z(2+i) - (2-i)z + (2+i)(2-i)$$

$$= z^2 - z(2+i+2-i) + 5 = \underline{z^2 - 4z + 5}$$

Polynomdiv: $z^3 - 11z + 20 \div z^2 - 4z + 5 = \underline{z+4}$

$-(z^3 - 4z^2 + 5z)$

$4z^2 - 16z + 20$

$-(4z^2 - 16z + 20)$

0

$\Downarrow$

Røttene er $\underline{\underline{2+i,\ 2-i}}$, -4

b) $\displaystyle\int \frac{10x+3}{x^3-11x+20}\,dx \underset{a)}{=} \int \frac{10x+3}{(x+4)\underbrace{(x^2-4x+5)}}\,dx$

Vet fra a) at denne ikke kan faktoriseres mer reelt

Delbrøksoppspaltning:

$$\frac{10x+3}{(x+4)(x^2-4x+5)} = \frac{A}{x+4} + \frac{Bx+C}{x^2-4x+5}$$

$$10x+3 = A(x^2-4x+5) + (Bx+C)(x+4)$$

$$= x^2(A+B) + x(-4A+4B+C) + (5A+4C)$$

$$\Downarrow$$

$A+B=0$, $-4A+4B+C=10$, $5A+4C=3$

$\Downarrow$ $A=-B \Rightarrow 8B+C=10$ $\Downarrow$ $C=10-8B \Rightarrow 3=-5B+40-32B$

$37B=37$

$\Downarrow$

$B=1$

$A=-1$, $C=2$

$$\int \frac{10x+3}{x^3-11x+20}\,dx = -\int \frac{1}{x+4}\,dx + \int \frac{x+2}{x^2-4x+5}\,dx$$

$$= -\ln|x+4| + \frac{1}{2}\int \frac{2x-4}{x^2-4x+5}\,dx + 4\int \frac{1}{x^2-4x+5}\,dx$$

$$= -\ln|x+4| + \frac{1}{2}\int \frac{1}{u}\,du + 4\int \frac{1}{(x-2)^2+1}\,dx$$

$u = x^2 - 4x + 5$

$du = (2x-4)\,dx$

$$= -\ln|x+4| + \frac{1}{2}\ln(x^2-4x+5) + 4\arctan(x-2) + C$$

$$\frac{1}{2}\int \frac{\cancel{2x-4}}{u}\,\frac{du}{\cancel{2x-4}}$$

$x^2 - 4x + 5$

$= x^2 - 4x + \left(\frac{4}{2}\right)^2 - \left(\frac{4}{2}\right)^2 + 5$

$= (x-2)^2 + 1$

31.) $\int \ln(x^2+2x+10)\,dx$

$$x\ln(x^2+2x+10) - \int \frac{2x^2+2x}{x^2+2x+10}\,dx$$

$u = \ln(x^2+2x+10)$

$v' = 1$

$\Downarrow$

$u' = \frac{1}{x^2+2x+10}(2x+2)$

$v = x$

Polynomdiv:

$2x^2 + 2x \div x^2 + 2x + 10 = 2 - \dfrac{2x+20}{x^2+2x+10}$

$-(2x^2 + 4x + 20)$

$-2x - 20$

M: $\displaystyle\int \frac{2x^2+2x}{x^2+2x+10}\,dx$:

$$\int \frac{2x^2+2x}{x^2+2x+10}\,dx = \int 2\,dx - \int \frac{2x+20}{x^2+2x+10}\,dx$$

$$= 2x - \int \frac{2x+2}{x^2+2x+10}\,dx - 18\int \frac{1}{x^2+2x+10}\,dx$$

$$= 2x - \int \frac{1}{u}\,du - \frac{18}{9}\int \frac{1}{\left(\frac{x+1}{3}\right)^2+1}\,dx$$

Substitusjon:

$u = x^2+2x+10$

$du = (2x+2)\,dx$

$\int \frac{\cancel{2x+2}}{u}\frac{du}{\cancel{2x+2}}$

$x^2+2x+10 = (x+1)^2+9$

$(x+1)^2+9 = 9\left(\frac{(x+1)^2}{9}+1\right)$

$= 9\left(\left(\frac{x+1}{3}\right)^2+1\right)$

$\frac{1}{1+\left(\frac{x+1}{3}\right)^2} \cdot \frac{1}{\cancel{3}}\cdot\cancel{3}$

$$= 2x - \ln(x^2+2x+10) - 2\arctan\left(\frac{x+1}{3}\right)3 + C$$

$$= 2x - 6\arctan\left(\frac{x+1}{3}\right) - \ln(x^2+2x+10) + C$$

Så:

$$\int \ln(x^2+2x+10)\,dx = x\ln(x^2+2x+10) + 6\arctan\left(\frac{x+1}{3}\right) + \ln(x^2+2x+10) - 2x + C$$