ℓ'Hopitals regel (6.3)

Teorem (ℓ'Hopital)

Hvis $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$, så er

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

gitt at grensen til høyre fins eller er $\pm\infty$.

Bevis Vi kan anta at $f(a) = g(a) = 0$ uten at grensene påvirkes. Cauchys middelverdisetning gir

$$[f(x) - f(a)] \cdot g'(c) = [g(x) - g(a)] \cdot f'(c)$$

for en c mellom a og x. Altså

$$f(x) \cdot g'(c) = g(x) \cdot f'(c)$$

$$\frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)}$$

Så

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(c)}{g'(c)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

siste likhet fordi c er mellom a og x. $\square$

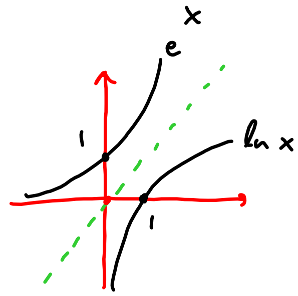
eks. 1

$$\lim_{x \to 0} \frac{e^x - 1}{x} \overset{\left[\frac{0}{0}\right]}{=} \lim_{x \to 0} \frac{e^x - 0}{1} = \frac{1}{1} = 1$$

eks. 2

$$\lim_{x \to 0} \frac{x^3 - x^2}{e^{2x} - 2e^x + 1} \overset{\left[\frac{0}{0}\right]}{=} \lim_{x \to 0} \frac{3x^2 - 2x}{2e^{2x} - 2e^x}$$

$$\overset{\left[\frac{0}{0}\right]}{=} \lim_{x \to 0} \frac{6x - 2}{4e^{2x} - 2e^x} = \frac{-2}{4-2} = -1$$



Varianter av l'Hopitals regel

① Broken $\frac{f(x)}{g(x)}$ kan gå mot $\left[\frac{\infty}{\infty}\right]$, $\left[\frac{-\infty}{\infty}\right]$ eller $\left[\frac{\infty}{-\infty}\right]$ istedenfor $\left[\frac{0}{0}\right]$

② Vi kan ha $x \to \infty$ eller $x \to -\infty$ istedenfor $x \to a$. Grensen kan også være ensidig.

Bevis Tar et eksempel. Anta at grensen er

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} \quad \text{der } f(x) \to 0 \text{ og } g(x) \to 0$$

Vi skifter variabel: $t = \frac{1}{x}$, dvs. $x = \frac{1}{t}$

$x \to \infty$ gir da $t \to 0^+$

Får:

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{t \to 0^+} \frac{f(1/t)}{g(1/t)} \overset{\left[\frac{0}{0}\right]}{=} \lim_{t \to 0^+} \frac{f'\left(\frac{1}{t}\right) \cdot \cancel{\left(-\frac{1}{t^2}\right)}}{g'\left(\frac{1}{t}\right) \cdot \cancel{\left(-\frac{1}{t^2}\right)}}$$

$$= \lim_{t \to 0^+} \frac{f'\left(\frac{1}{t}\right)}{g'\left(\frac{1}{t}\right)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)} \quad \square$$

eks. 1

$$\lim_{x \to \infty} \frac{6x^2 + 3x}{2x^2 + 1} \overset{\left[\frac{\infty}{\infty}\right]}{=} \lim_{x \to \infty} \frac{12x + 3}{4x}$$

$$\overset{\left[\frac{\infty}{\infty}\right]}{=} \lim_{x \to \infty} \frac{12}{4} = \underline{\underline{3}}$$

eks. 2

$$\lim_{x \to 0^+} \frac{1 + \frac{1}{x}}{\ln x} \overset{\left[\frac{\infty}{-\infty}\right]}{=} \lim_{x \to 0^+} \frac{-\frac{1}{x^2} \cdot x}{\frac{1}{x} \cdot x}$$

$$= \lim_{x \to 0^+} \frac{-\frac{1}{x}}{1} = \underline{\underline{-\infty}}$$

$\boxed{\text{Trikseformer}}$

Formen $[0 \cdot \infty]$

Metode: Skriv $f(x) \cdot g(x)$ som $\dfrac{f(x)}{\frac{1}{g(x)}}$ eller $\dfrac{g(x)}{\frac{1}{f(x)}}$

og bruk vanlig l'Hopital

eks. $\lim_{x \to 0^+} x \ln x \overset{[0\cdot\infty]}{=} \lim_{x \to 0^+} \dfrac{\ln x}{\frac{1}{x}}$

$\overset{\left[\frac{\infty}{\infty}\right]}{=} \lim_{x \to 0^+} \dfrac{\frac{1}{x} \cdot x^2}{-\frac{1}{x^2} \cdot x^2} = \lim_{x \to 0^+} \dfrac{x}{-1} = \underline{\underline{0}}$

Formene $[1^\infty]$, $[0^0]$ og $[\infty^0]$

Skriv om ved hjelp av

$$a^b = \left(e^{\ln a}\right)^b = e^{(\ln a)\cdot b}$$

og bruk l'Hopital på eksponenten.

eks. 1 $\lim\limits_{x\to 0^+} (3x)^x \overset{[0^0]}{=} \lim\limits_{x\to 0^+} e^{(\ln 3x)\cdot x}$

Eksponent:

$$\lim_{x\to 0^+} (\ln 3x)\cdot x \overset{[\infty\cdot 0]}{=} \lim_{x\to 0^+} \frac{\ln(3x)}{\frac{1}{x}}$$

$$\overset{[\frac{\infty}{\infty}]}{=} \lim_{x\to 0^+} \frac{\frac{1}{3x}\cdot 3}{\frac{-1}{x^2}} = \lim_{x\to 0^+} \frac{x}{-1} = 0.$$

Så $\lim\limits_{x\to 0^+} (3x)^x = e^0 = \underline{1}$

eks. 2 $\lim\limits_{x\to 0} (1-3x^2)^{1/x^2} \overset{[1^\infty]}{=} \lim\limits_{x\to 0} e^{[\ln(1-3x^2)]\cdot\frac{1}{x^2}}$

Eksponent:

$$\lim_{x\to 0} [\ln(1-3x^2)]\cdot\frac{1}{x^2} \overset{[0\cdot\infty]}{=} \lim_{x\to 0} \frac{\ln(1-3x^2)}{x^2}$$

$$\overset{[\frac{0}{0}]}{=} \lim_{x\to 0} \frac{\frac{1}{1-3x^2}\cdot(-6\cancel{x})}{2\cancel{x}}$$

$$= -3\cdot \lim_{x\to 0} \frac{1}{1-\underbrace{3x^2}_{\to 0}} = -3\cdot 1 = -3$$

Så $\lim\limits_{x\to 0} (1-3x^2)^{1/x^2} = \underline{e^{-3}}$