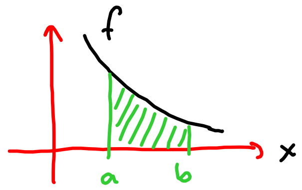
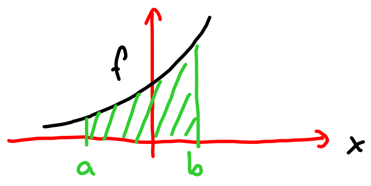
Uegentlige (uekte) integraler (9.5)

$$\int_a^\infty f(x)\,dx \overset{\text{def}}{=} \lim_{b\to\infty} \int_a^b f(x)\,dx$$



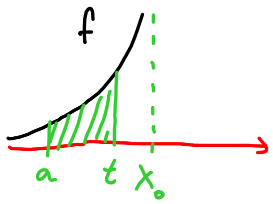
$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to -\infty} \int_a^b f(x)\,dx$$



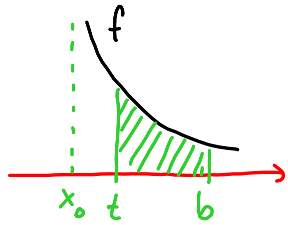
$$\int_{-\infty}^\infty f(x)\,dx \overset{\text{def}}{=} \int_{-\infty}^0 f(x)\,dx + \int_0^\infty f(x)\,dx$$

Hvis f har en vertikal asymptote i $x = x_0$, definerer vi

$$\int_a^{x_0} f(x)\,dx \overset{\text{def}}{=} \lim_{t\to x_0^-} \int_a^t f(x)\,dx \quad \text{for } a < x_0$$

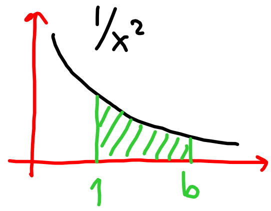


$$\int_{x_0}^b f(x)\,dx \overset{\text{def}}{=} \lim_{t\to x_0^+} \int_t^b f(x)\,dx \quad \text{for } b > x_0$$



<u>eks. 1</u>

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx$$

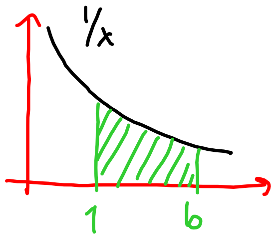


$$= \lim_{b\to\infty} \left[-\frac{1}{x}\right]_1^b$$

$$= \lim_{b\to\infty} \left[-\frac{1}{b} - \left(-\frac{1}{1}\right)\right] = \lim_{b\to\infty} \left[1 - \frac{1}{b}\right] = \underline{\underline{1}}. \quad \square$$

<u>eks. 2</u>

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx$$

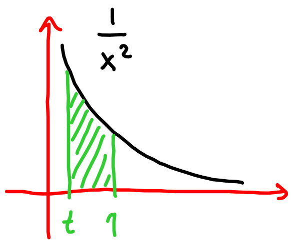


$$= \lim_{b\to\infty} \Big[\ln|x|\Big]_1^b = \lim_{b\to\infty} \Big[\ln b - \overset{0}{\ln 1}\Big]$$

$$= \underline{\underline{+\infty}}$$

<u>eks. 3</u>

$$\int_0^1 \frac{1}{x^2}\,dx = \lim_{t\to 0^+} \int_t^1 \frac{1}{x^2}\,dx$$



$$= \lim_{t\to 0^+} \left[-\frac{1}{x}\right]_t^1$$

$$= \lim_{t\to 0^+} \left[-\frac{1}{1} - \left(-\frac{1}{t}\right)\right] = \lim_{t\to 0^+} \left[\overset{+\infty}{\frac{1}{t}} - 1\right] = \underline{\underline{+\infty}}$$

(divergerer)

Hvis vi får et endelig tall som svar på et uekte integral, sier vi at integralet <u>konvergerer</u>. I motsatt fall sier vi at integralet <u>divergerer</u>.

Teorem (p-integralene)

Integralet $\int_1^\infty \frac{1}{x^p}\,dx$ konvergerer for $p > 1$, og divergerer for $p \leq 1$.

Bevis Kan anta $p \neq 1$, siden vi vet at det gir divergens. Får

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{b\to\infty} \int_1^b x^{-p}\,dx$$

$$= \lim_{b\to\infty} \left[\frac{1}{-p+1} x^{-p+1}\right]_1^b = \lim_{b\to\infty}\left[\frac{1}{1-p} b^{1-p} - \frac{1}{1-p}\cdot 1\right]$$

$$= \frac{1}{1-p} \lim_{b\to\infty}\left[b^{1-p} - 1\right]$$

Her går b^{1-p} mot $+\infty$ hvis $p < 1$, og mot 0 hvis $p > 1$. $\square$

Teorem (Sammenlikningstesten for integraler)

La f og g være kontinuerlige med $0 \leq g(x) \leq f(x)$ for alle $x > a$. Da:

(i) Hvis $\int_a^\infty f(x)\,dx$ konv., så konv. også $\int_a^\infty g(x)\,dx$.

(ii) Hvis $\int_a^\infty g(x)\,dx$ div., så div. også $\int_a^\infty f(x)\,dx$

Bevis Se bok. $\square$

eks. $\displaystyle\int_1^\infty \frac{1}{x^2+\sin^2 x}\,dx$ Konvergerer dette?

Vi har

$$0 \leq \underbrace{\frac{1}{x^2+\sin^2 x}}_{g(x)} \leq \underbrace{\frac{1}{x^2}}_{f(x)}$$

Bruker sammenlikn. testen punkt (i)

Vi vet at $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ konvergerer (p-integral, $p=2$)

Ergo konvergerer integralet vårt ved sammenlikningstesten. $\square$

Teorem (Grense-sammenlikningstesten for integraler)

Gitt integralene $\int_a^\infty f(x)\,dx$ og $\int_a^\infty g(x)\,dx$, der f og g er positive og kontinuerlige. Hvis

$$L = \lim_{x\to\infty} \frac{f(x)}{g(x)} \quad \text{fins og} \quad 0 < L < \infty$$

så enten konvergerer begge integralene eller divergerer begge integralene.

Bevis Velg konstanter P og Q slik at $0 < P < L < Q$.
Siden $f(x)/g(x) \to L$, har vi for tilstrekkelig store x

$$P < \frac{f(x)}{g(x)} < Q, \quad \text{dvs.} \quad P\cdot g(x) < f(x) < Q\cdot g(x).$$

Vi bruker så den vanlige sammenlikningstesten:

$\int f(x)dx$ konv $\Rightarrow \int P\cdot g(x)\,dx$ konv., dvs. $\int g(x)dx$ konv.

$\int f(x)dx$ div $\Rightarrow \int Q\cdot g(x)\,dx$ div, dvs. $\int g(x)dx$ div. $\square$

eks. $\displaystyle\int_1^\infty \frac{x^{3/2}+1}{x^{5/2}+1}\,dx$ Konvergerer dette?

Funksjonen "går som" $\dfrac{x^{3/2}}{x^{5/2}} = \dfrac{1}{x}$ (gir divergent p-integral)

Vi sammenlikner derfor med $g(x) = \frac{1}{x}$ i grensesammenlikningstesten:

$$L = \lim_{x\to\infty} \frac{\left(\frac{x^{3/2}+1}{x^{5/2}+1}\right)\cdot x}{\left(\frac{1}{x}\right)\cdot x} = \lim_{x\to\infty} \frac{x^{5/2}+x}{x^{5/2}+1}$$

$$= \lim_{x\to\infty} \frac{1 + \overbrace{\left(\frac{x}{x^{5/2}}\right)}^{\to 0}}{1 + \underbrace{\left(\frac{1}{x^{5/2}}\right)}_{\to 0}} = 1.$$

Altså divergerer integralet vårt ved grense-sml.-testen, fordi vi vet at

$\displaystyle\int_1^\infty \frac{1}{x}\,dx$ divergerer. $\square$