SNUBLEGRUPPE MAT 1100 21.09.2016

$\{a_n\}_{n=0}^{\infty} = \{a_0, a_1, a_2, \dots\}$

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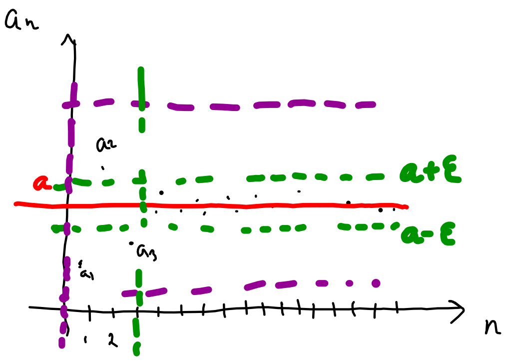
$a_n \to a$

$\lim_{n\to\infty} a_n = a$

$\{a_n\}$ konvergerer mot a hvis:

Gitt $\varepsilon > 0$ $\exists$ (Eksisterer) $N \in \mathbb{N}$ s.a. $|a_n - a| < \varepsilon$ $\forall$ (for alle) $n \geq N$



$N=4$

$N=1$

$a_n \to a$
$b_n \to b$

$\pm$ $\lim_{n\to\infty}(a_n \pm b_n) = a \pm b$

$\cdot$ $\lim_{n\to\infty}(a_n \cdot b_n) = a \cdot b$

$\div$ $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{a}{b}$

hvis $b \neq 0$

VIKTIG TEOREM: En monoton begrenset følge er alltid konvergent

TRIKS: BEREGNING AV GRENSEVERDIER

BRØK • Utvide med $\frac{1}{\text{høyeste potens}}$

RØTTER OG LEDD • Utvide med "den konjugerte"

NB! • $\lim_{n\to\infty} \frac{1}{n} = 0$

Gitt $\varepsilon > 0$

$$\left|\frac{1}{n} - 0\right| = \left|\frac{1}{n}\right| = \frac{1}{n} < \varepsilon \quad \text{for alle } n \geq N$$

Velg N s.a. $\frac{1}{N} < \varepsilon$ dvs jeg velger $N > \frac{1}{\varepsilon}$

4.3.3. a) $\lim_{n\to\infty}\left(\sqrt{n+2}-\sqrt{n}\right) = \lim_{n\to\infty}\frac{\left(\sqrt{n+2}-\sqrt{n}\right)\cdot\left(\sqrt{n+2}+\sqrt{n}\right)}{\left(\sqrt{n+2}+\sqrt{n}\right)}$

$(a-b)(a+b)=a^2-b^2$

$$= \lim_{n\to\infty}\frac{n+2-n}{\left(\sqrt{n+2}+\sqrt{n}\right)}$$

$$= \lim_{n\to\infty}\frac{2}{\left(\sqrt{n+2}+\sqrt{n}\right)}$$

$$= \underline{\underline{0}}$$

4.3.3. d) $\lim_{n\to\infty} \left(\sqrt{1+e^{-2n}} - e^{-n}\right) = " \left(\sqrt{1+0} - 0\right) " = \underline{\underline{1}}$

$$= \lim_{n\to\infty} \frac{\left(\sqrt{1+e^{-2n}} - e^{-n}\right)\left(\sqrt{1+e^{-2n}} + e^{-n}\right)}{\left(\sqrt{1+e^{-2n}} + e^{-n}\right)}$$

$$= \lim_{n\to\infty} \frac{1+e^{-2n} - e^{-2n}}{\left(\sqrt{1+e^{-2n}} + e^{-n}\right)}$$

$$= \lim_{n\to\infty} \frac{1}{\left(\sqrt{1+e^{-2n}} + e^{-n}\right)} = \underline{\underline{1}}$$

4.3.4. b) $\lim_{n \to \infty} \frac{2 \sin n}{n} = 0$, $\left|\frac{2\sin n}{n}\right| = \left|\frac{2}{n}\right| |\sin n| \leq \left|\frac{2}{n}\right| \cdot 1$

Gitt $\varepsilon > 0$, $\left| \frac{2 \sin n}{n} - 0 \right| \leq \left| \frac{2}{n} \right| = \frac{2}{n}$

Velg N s.a. $\frac{2}{N} < \varepsilon$ dvs. $N > \frac{2}{\varepsilon}$.

Da er $\left| \frac{2 \sin n}{n} - 0 \right| \leq \frac{2}{N} < \frac{2}{\frac{2}{\varepsilon}} = \varepsilon \quad \forall\ n \geq N$

c) $\lim_{n\to\infty} \frac{n+\frac{1}{2}}{3n+2} = \frac{1}{3}$

Gitt $\varepsilon > 0$, $\left|\frac{n+\frac{1}{2}}{3n+2} - \frac{1}{3}\right| = \left|\frac{3(n+\frac{1}{2}) - (3n+2)}{3(3n+2)}\right|$

$= \left|\frac{\cancel{3}n + \frac{3}{2} - \cancel{3}n - 2}{3(3n+2)}\right| = \left|\frac{-\frac{1}{2}}{3(3n+2)}\right|$

$= \frac{1}{18n+12} \leq \frac{1}{18n}$

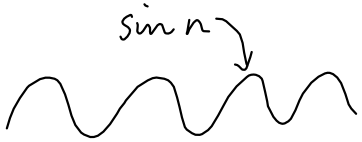
Velg N s.a. $\frac{1}{18N} < \varepsilon$ dvs $N > \frac{1}{18\varepsilon}$

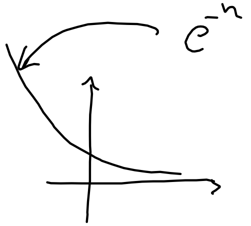
så er $\left|\frac{n+\frac{1}{2}}{3n+2} - \frac{1}{3}\right| < \varepsilon \quad \forall \; n \geq N.$

4.3.1.

a) $$\lim_{n\to\infty} \frac{8n^4+2n}{3n^4-7} = \lim_{n\to\infty} \frac{\frac{1}{n^4}\cdot(8n^4+2n)}{\frac{1}{n^4}\cdot(3n^4-7)} = \lim_{n\to\infty} \frac{\frac{8\cancel{n^4}}{\cancel{n^4}} + \frac{2\cancel{n}}{n^{\cancel{4}3}}}{\frac{3\cancel{n^4}}{\cancel{n^4}} - \frac{7}{n^4}} = \lim_{n\to\infty} \frac{8 + \frac{2}{n^3}\ 0}{3 - \frac{7}{n^4}} = \underline{\underline{\frac{8}{3}}}$$

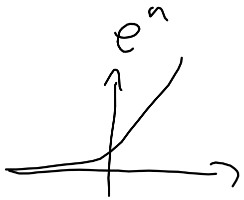
b) $$\lim_{n\to\infty} \frac{n^5 + 2\sin n}{e^{-n}+6n^5} = \lim_{n\to\infty} \frac{\frac{1}{n^5}(n^5+2\sin n)}{\frac{1}{n^5}(e^{-n}+6n^5)} = \lim_{n\to\infty} \frac{1 + \frac{2\sin n}{n^5}}{\frac{e^{-n}}{n^5} + 6}$$





$$= \frac{1+0}{0+6}$$ (siden $|2\sin n| \le 2$) (siden $\lim_{n\to\infty} e^{-n} = 0$)

$$= \frac{1}{6}$$



4.3.18. $\{x_n\}$ $x_1 = 1$ $x_{n+1} = \sqrt{2x_n}$ for $n \geq 1$

a) Vis ved induksjon på n at $x_n < x_{n+1}$ for alle $n \in \mathbb{N}$

① • $x_1 = 1 < x_2 = \sqrt{2 \cdot x_1} = \sqrt{2}$

② • Anta $x_{n-1} < x_n$

③ • $2x_{n-1} < 2x_n$

$$x_n = \sqrt{2x_{n-1}} < \sqrt{2x_n} = x_{n+1}$$

Så ved induksjon er $x_n < x_{n+1}$ $\forall$ $n \in \mathbb{N}$.

$\{x_n\}$ er en monotont voksende følge!

b) Gjenstår å vise $\{x_n\}$ begrenset.

- Anta at det finnes en grense.

$$\lim_{n\to\infty} x_n = A$$

$$A = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \sqrt{2x_n} = \sqrt{2A}$$

$$A = \sqrt{2A}$$

$$A^2 = 2A$$

$$A^2 - 2A = 0$$

$$A(A-2) = 0$$

$$\cancel{A=0} \wedge \underline{A = 2}$$

Viser $\{x_n\}$ begrenset.

Induksjon:

- $x_1 = 1 < 2$

  $(x_2 = \sqrt{2} < 2)$

- Anta $x_n < 2$
- Må vise $x_{n+1} < 2$

$$x_{n+1} = \sqrt{2x_n} < \sqrt{2 \cdot 2} = 2.$$

KONKLUSJON: Vi har en monoton, begrenset følge som konvergerer mot 2.

c) $\{y_n\}$ $\underline{y_1 = 1}$ $y_{n+1} = \sqrt{2y_n + y_n^2}$ for $n \geq 1$

MONOTON

- $y_1 = 1 < y_2 = \sqrt{2 \cdot 1 + 1^2} = \sqrt{3}$
- Anta $y_{n-1} < y_n$
- $2y_{n-1} < 2y_n$

$$y_n = \sqrt{2y_{n-1} + y_{n-1}^2} < \sqrt{2y_n + y_n^2} = y_{n+1} \quad (\text{siden } y_{n-1} < y_n)$$

ok v/ induksjon.

MULIG GRENSE: $\lim_{n\to\infty} y_n = A = \lim_{n\to\infty} y_{n+1}$

$$A = \sqrt{2A + A^2}$$

$$0 \ \cancel{A^2} = 2A + \cancel{A^2}$$

$$\underline{A = 0} \quad \underline{\underline{\text{UMULIG}}} \text{ fordi } y_1 = 1 \text{ og } y_n < y_{n+1}$$