FUNKSJONER FRA $\mathbb{R}^n \longrightarrow \mathbb{R}^m$

- Partiell derivasjon
- $f : \underset{\mathbb{R}^n}{\overset{\cap\!\shortparallel}{A}} \longrightarrow \mathbb{R}$ $\quad \underbrace{f'(\vec{a}, \vec{r})}_{\text{RETNINGSDERIVERT}} = \nabla f(\vec{a}) \cdot \vec{r}$ (f deriverbar i $\vec{a}$)
- $\nabla f(\vec{a})$ peker i den retningen der f vokser raskest i $\vec{a}$.

H 2012

(2) $f(x,y) = x^5y^7 - 4x^2 + y$

Vi skal finne $\frac{\partial f}{\partial x}$:

$$\frac{\partial f}{\partial x} = y^7 \cdot 5x^4 - 8x + 0$$
$$= \underline{5x^4y^7 - 8x}$$

(4) I punktet $\vec{a} = (0,1,5)$, skal vi finne hvilken retning $\boxed{f(x,y,z) = e^{2xy} - \frac{1}{2}e^{xz}}$ vokser raskest.

- Funksjonen f vokser raskest i retning $\nabla f(\vec{a})$.

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

$$\frac{\partial f}{\partial x} = e^{2xy} \cdot 2y - \frac{1}{2}e^{xz} \cdot z$$

$$\frac{\partial f}{\partial y} = e^{2xy} \cdot 2x$$

$$\frac{\partial f}{\partial z} = -\frac{1}{2}e^{xz} \cdot x$$

$$\frac{\partial f}{\partial x}(0,1,5) = e^{2\cdot 0\cdot 1} \cdot 2\cdot 1 - \frac{1}{2}e^{0\cdot 5} \cdot 5 = 2 - \frac{5}{2} = \underline{-\frac{1}{2}}$$

$$\frac{\partial f}{\partial y}(0,1,5) = e^{2\cdot 0\cdot 1} \cdot 2\cdot 0 = \underline{0}$$

$$\frac{\partial f}{\partial z}(0,1,5) = -\frac{1}{2}e^{0\cdot 5} \cdot 0 = \underline{0}$$

$$\underline{\nabla f(0,1,5) = \left(-\frac{1}{2}, 0, 0\right)}$$

<u>Funksjonen f vokser i $(0,1,5)$ raskest i retning $(-\frac{1}{2}, 0, 0)$</u>

5) $f(x,y) = x^2y - xy^2$ $\quad \mathbb{R}^2 \longrightarrow \mathbb{R}$

Skal finne den retningsderiverte $f'(\vec{a},\vec{r})$
der $\vec{a} = (4,1)$ og $\vec{r} = (1,1)$.

Vi vet at $f'(\vec{a},\vec{r}) = \nabla f(\vec{a}) \cdot \vec{r}$

Må finne $\nabla f(\vec{a})$:

$\frac{\partial f}{\partial x} = 2xy - y^2$ $\qquad \frac{\partial f}{\partial x}(4,1) = 2\cdot 4\cdot 1 - 1^2 = \underline{7}$

$\frac{\partial f}{\partial y} = x^2 - 2xy$ $\qquad \frac{\partial f}{\partial y}(4,1) = 4^2 - 2\cdot 4\cdot 1 = \underline{8}$

$\nabla f(4,1) = (7,8)$

$f'(\vec{a};\vec{r}) = \nabla f(4,1)\cdot\vec{r} = (7,8)\cdot(1,1) = 7+8 = \underline{\underline{15}}$

H 2013

① $f(x,y,z) = z + \arctan(xy+1)$

Vi skal finne $\frac{\partial f}{\partial x}$:

$$\frac{\partial f}{\partial x} = \frac{1}{1+(xy+1)^2}\,\frac{\partial(xy+1)}{\partial x}$$

$$= \underline{\underline{\frac{y}{1+(xy+1)^2}}}$$

② $\vec{r} = (1,0)$ $\quad \vec{a} = (1,1)$

Skal bestemme hvilken funksjon som har retningsderivert $f'(\vec{a};\vec{r}) = 0$.

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \| $

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \nabla f(\vec{a})\cdot\vec{r}$

Ⓐ $f(x,y) = x$

$\nabla f = (1,0)$ (for alle $\vec{a}$)

$\nabla f(1,1)\cdot\vec{r} = (1,0)\cdot(1,0) = 1 \neq 0$.

Det er ikke denne funksjonen!

Ⓑ $f(x,y) = (x-1)^2 + y^2$

$\nabla f = (2(x-1),\ 2y)$

$\nabla f(\overset{\vec{a}}{1,1}) = (0,2)$

$\nabla f(1,1)\cdot(1,0) = (0,2)\cdot(1,0)$

$= \underline{\underline{0}}$

Så B er riktig funksjon!

H 2014 ① $f(x,y) = x\sin(xy^2)$

Skal finne $\frac{\partial f}{\partial y}$:

$$\frac{\partial f}{\partial y} = x\cos(xy^2)\cdot 2xy$$
$$= \underline{\underline{2x^2y\cos(xy^2)}}$$

② $f(x,y) = xe^{xy}$ Skal finne $f'(\vec{a};\vec{r})$ der $\vec{a} = (1,1)$ og $\vec{r} = (-2,1)$

$$f'(\vec{a};\vec{r}) = \nabla f(\vec{a})\cdot\vec{r}$$

Finner først $\nabla f(\vec{a})$:

$$\nabla f = \left(1\cdot e^{xy} + xe^{xy}\cdot y\ ,\ xe^{xy}\cdot x\right)$$
$$= \left(e^{xy}(1+xy)\ ,\ x^2e^{xy}\right)$$
$$\nabla f(1,1) = \left(e^{1\cdot 1}(1+1\cdot 1)\ ,\ 1^2\cdot e^{1\cdot 1}\right)$$
$$= \underline{(2e\ ,\ e)}$$

$$f'(\vec{a};\vec{r}) = \nabla f(\vec{a})\cdot\vec{r} = (2e,e)\cdot(-2,1)$$
$$= -4e + e$$
$$= \underline{\underline{-3e}}$$

V 2008

del 2 - OPPG. 1.

a) $P(z) = z^3 + 8$

Skal finne reell og kompleks faktorisering.

- Observerer at $P(-2) = (-2)^3 + 8 = 0$, så $(z+2)$ er en faktor.
- Gjør polynomdivisjon

$$
\begin{array}{l}
z^3 + 8 : z + 2 = \underline{z^2 - 2z + 4} \\
\underline{-(z^3 + 2z^2)} \\
\quad -2z^2 + 8 \\
\underline{-(-2z^2 - 4z)} \\
\quad 4z + 8 \\
\underline{-(4z + 8)} \\
\quad 0
\end{array}
$$

- Sjekker om $z^2 - 2z + 4$ har reelle eller komplekse røtter og finner dem.

$$
\begin{aligned}
z &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot 4}}{2} \\
&= \frac{2 \pm \sqrt{-12}}{2} \\
&= \frac{2 \pm 2i\sqrt{3}}{2} \\
&= \underline{1 \pm i\sqrt{3}}
\end{aligned}
$$

Kompleks faktorisering: $P(z) = (z+2)(z - (1 + i\sqrt{3}))(z - (1 - i\sqrt{3}))$

Reell faktorisering: $P(z) = (z+2)(z^2 - 2z + 4)$

b) $\dfrac{12}{(x+2)(x^2-2x+4)} = \dfrac{A}{x+2} + \dfrac{Bx+C}{x^2-2x+4}$

$= \dfrac{A(x^2-2x+4) + (Bx+C)(x+2)}{\text{Fellesnevner}}$

$12 = Ax^2 - 2Ax + 4A + Bx^2 + 2Bx + Cx + 2C$

$$\begin{cases} 0 = A + B & (x^2) \\ 0 = -2A + 2B + C & (x) \\ 12 = 4A + 2C & (x^0) \end{cases}$$

$B = -A$

$-4A + C = 0$ $\quad C = 4A$

$12 = 4A + 2 \cdot 4A$

$\underline{A = 1}$

$\underline{C = 4}$

$\underline{B = -1}$

$$\underline{\underline{\frac{12}{(x+2)(x^2-2x+4)} = \frac{1}{x+2} + \frac{-x+4}{x^2-2x+4}}}$$

c) $\int \frac{x-4}{x^2-2x+4}\,dx$

Shrive om:

$\frac{1}{2}\int \frac{2x-8}{x^2-2x+4}\,dx$

$= \frac{1}{2}\int \frac{2x-2}{x^2-2x+4}\,dx - \int \frac{3}{x^2-2x+4}\,dx$

$= \frac{1}{2}\int \frac{1}{u}\,du - 3\int \frac{1}{(x-1)^2+3}\,dx$

$= \frac{1}{2}\ln(\underbrace{x^2-2x+4}_{>0}) - \frac{3}{3}\int \frac{1}{\left(\frac{x-1}{\sqrt{3}}\right)^2+1}\,dx$

$= \frac{1}{2}\ln(x^2-2x+4) - \sqrt{3}\arctan\left(\frac{x-1}{\sqrt{3}}\right) + C$

SUBST:

$u = x^2-2x+4$

$du = 2x-2$

$\underbrace{x^2-2x+1}_{(x-1)^2} + 3$

SUBST

$u = \left(\frac{x-1}{\sqrt{3}}\right)$

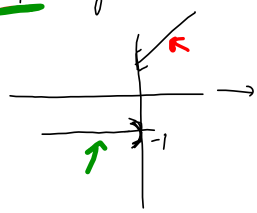
$du = \frac{1}{\sqrt{3}}\,dx$

$\sqrt{3}\,du = dx$

$\int \frac{1}{u^2+1}\cdot\sqrt{3}\,du$

H 2011 (13) $f: (0,\infty) \longrightarrow \mathbb{R}$

$$f = \begin{cases} \dfrac{\ln x}{x-1} & \text{for } x \neq 1 \\ 1 & \text{for } x = 1 \end{cases} \qquad g = \begin{cases} x+1 & \text{for } x \geq 0 \\ -1 & \text{for } x < 0 \end{cases}$$



a) Skal vise at f er kontinuerlig.

- For $x \neq 1$ er f en brøk av to kontinuerlige funksjoner, og f er derfor kontinuerlig for $x \neq 1$.
- For $x = 1$: f er kontinuerlig i $x = 1$ hvis

$$\lim_{x \to 1^-} f(x) = \underset{\substack{\shortparallel \\ 1}}{f(1)} = \lim_{x \to 1^+} f(x)$$

$$\lim_{x \to 1^-} \frac{\ln x}{x-1} \overset{\text{"}\frac{0}{0}\text{"}}{\underset{\text{L'H}}{=}} \lim_{x \to 1^-} \frac{\frac{1}{x}}{1} = \underline{\underline{1}}$$

Tilsvarende for $\lim\limits_{x \to 1^+}$.

- Dermed er f kontinuerlig på $(0,\infty)$.

b) 1) finne den deriverte til f når $x \neq 1$.

$$f(x) = \frac{\ln x}{x-1}$$

$$f'(x) = \frac{\frac{1}{x}(x-1) - \ln x \cdot 1}{(x-1)^2}$$

$$= \frac{1 - \frac{1}{x} - \ln x}{(x-1)^2}$$

2) Skal vise at f er deriverbar i $x=1$ og finne $f'(1)$ (hvis den eksisterer)

v/a) er f kontinuerlig.

Nok å vise at $\lim\limits_{x \to 1} \frac{f(x) - f(1)}{x-1}$ eksisterer

$$\lim_{x \to 1^-} \frac{\frac{\ln x}{x-1} - 1}{x - 1} = \lim_{x \to 1^-} \frac{\ln x - x + 1}{(x-1)^2} \underset{L'H}{\overset{"\frac{0}{0}"}{=}} \lim_{x \to 1^-} \frac{\frac{1}{x} - 1}{2(x-1) \cdot 1}$$

$$\underset{L'H}{\overset{"\frac{0}{0}"}{=}} \lim_{x \to 1^-} \frac{-\frac{1}{x^2}}{2} = -\frac{1}{2}$$

Nøyaktig samme for $\lim\limits_{x \to 1^+}$, så f er deriverbar i $x=1$

og $f'(1) = -\frac{1}{2}$