Anvendelser av integralet

- areal
- volum
- lengder
- (arbeid).

$\Pi = \{x_0, x_1, \dots, x_n\}$ partisjon av $[a,b]$

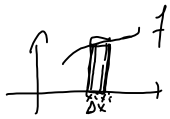
$$\sup_{\Pi} N(\Pi) = \inf_{\Pi} \phi(\Pi)$$

$$\sum_{i=1}^{n} m_i (x_i - x_{i-1}) \qquad \sum_{i=1}^{n} M_i (x_i - x_{i-1})$$

$$m_i = f(c_i), \quad x_i - x_{i-1} = \Delta x_i, \quad \int_a^b f(x)\,dx, \quad M_i = f(c_i')$$

$$\sup N(\Pi) = \sup\left(\sum f(c_i)\,\Delta x_i\right) \le \int_a^b f(x)\,dx \le \inf \phi(\Pi) = \inf \sum f(c_i')\,\Delta x_i$$

$$\sum \le \int \le \sum$$



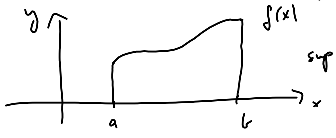
$$\sum_{c} f(c_i)\,\Delta x \le \int_a^b f(x)\,dx \le \sum f(c_i')\,\Delta x$$

$$c_i = x_i, \quad c_i' = x_i$$

$$f(x_i)\,\Delta x \qquad f(x)\,dx \qquad f(x_i')\,\Delta x$$

$$\sum_{|\Pi| \to 0} f(x_i)\,\Delta x \longrightarrow \int_a^b f(x)\,dx$$

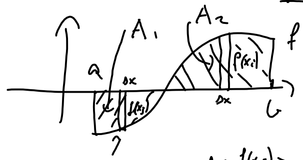
Dersom $f : [a,b]$ er integrerbar og $f(x) \ge 0$ for hver $x \in [a,b]$, så er $\int_a^b f(x)\,dx$ lik arealet under grafen til f.



$$\sup N(\Pi) = \int_a^b f(x)\,dx = \inf \phi(\Pi)$$

$\le A \le$, A = arealet under grafen

Hva med

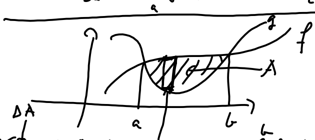


$$\int_a^b f(x)\,dx \overset{?}{=}$$

$\Delta x f(x_j) < 0$
$= -($arealet til $\Box_{\Delta x} -f(x_j))$

$\Delta x \cdot f(x_i) > 0$
$=$ arealet til $\Box_{\Delta x} f(x_i)$

Så $\int_a^b f(x)\,dx = A_2 - A_1$



$$A = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

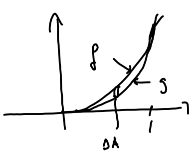
$f(x_i) - g(x_i)\{\Box_{\Delta x}$

$$\lim_{|\Pi| \to 0} \sum_{\Pi} (f(x_i) - g(x_i))\,\Delta x = \int_a^b (f(x) - g(x))\,dx$$

$$(f(x_i) - g(x_i))\,\Delta x \qquad (f(x) - g(x))\,dx$$

$$A = \lim_{|\Pi| \to 0} \sum \Delta A \qquad \lim_{|\Pi| \to 0} \sum \Delta A = \int dA$$

Ex: $f(x) = x^2 \quad g(x) = x^3 \quad$ på $[0,1]$



$$\Delta A = (f(x_i) - g(x_i))\,\Delta x = (x_i^2 - x_i^3)\,\Delta x$$

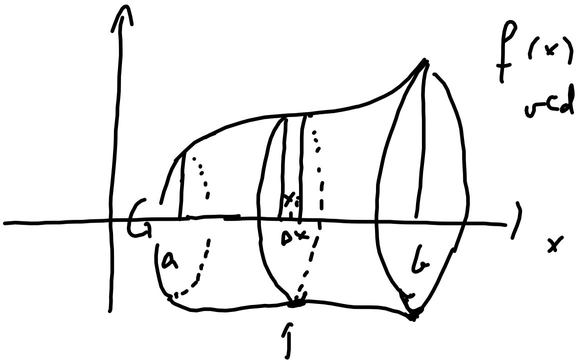
$|\Pi| \to 0$:

$$dA = (f(x) - g(x))\,dx = (x^2 - x^3)\,dx$$

$$\underline{A} = \int dA = \int_0^1 (x^2 - x^3)\,dx$$

$$= \left[\tfrac{1}{3}x^3 - \tfrac{1}{4}x^4\right]_0^1 = \tfrac{1}{3}\cdot 1^3 - \tfrac{1}{4}\cdot 1^4 - (0 - 0) = \tfrac{1}{3} - \tfrac{1}{4} = \underline{\tfrac{1}{12}}$$

Volum av omdreiningslegeme.



Vil finne volumet V til omdreiningslegemet vi får ved å dreie grafen til f om x-aksen.

$$\underline{V} = \lim_{|\Pi|\to 0} \sum \Delta V$$

$$\underline{\Delta V} = V(\text{disk: } f(x_i),\ \Delta x) = \pi f(x_i)^2 \cdot \Delta x$$

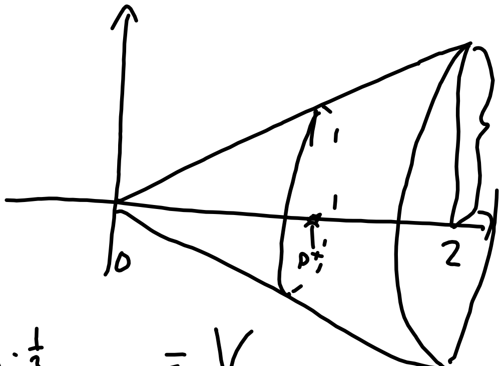
$$= \lim_{|\Pi|\to 0} \sum \pi f(x_i)^2 \Delta x$$

$$= \int_a^b \pi f(x)^2 \, dx$$

siden $\pi f(x)^2$ er integrerbar når f er det (på $[a,b]$).

$$= \int_a^b dV$$

$$\underline{dV = \pi f(x)^2 dx}$$



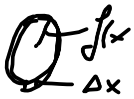
$f(x) = \frac{1}{2} x \qquad x \in [0,2]$

Volumet til kjeglen en får ved å dreie grafen til f om x-aksen er

$$G \cdot h \cdot \tfrac{1}{3} = \pi \cdot 1^2 \cdot 2 \cdot \tfrac{1}{3} = \frac{2\pi}{3}$$

$$= V = \int dV = \lim_{|\Pi|\to 0} \sum \pi \left(\tfrac{1}{2} x_i\right)^2 \Delta x \overset{\text{def}}{=} \lim_{|\Pi|\to 0} \sum \Delta V$$

$$= \int_0^2 \pi \left(\tfrac{1}{2}\right)^2 x^2 \cdot dx =$$



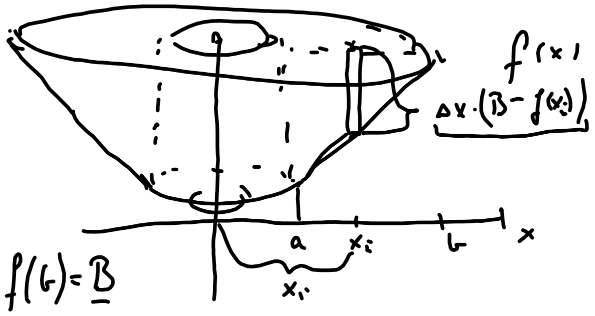
$$V = \int dV = \lim_{|\Pi|\to 0} \sum \Delta V$$

$$= \lim \sum \pi f(x_i)^2 \cdot \Delta x$$

$$= \int_a^b \pi f^2 dx \qquad \underline{dV = \pi f^2 dx}$$

$$= \left[\pi \cdot \left(\tfrac{1}{2}\right)^2 \cdot \tfrac{1}{3} x^3\right]_0^2 = \frac{\pi}{4} \cdot \frac{1}{3} \cdot (8 - 0)$$

$$= \frac{2\pi}{3}$$



Vil finne volumet V til omdreiningslegemet som en får ved å dreie grafen til f om y aksen

$f(b) = B$

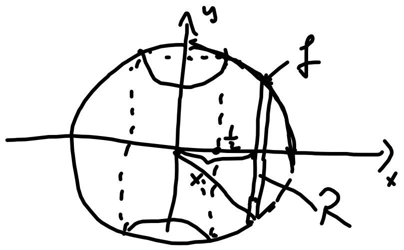
$$\underline{V} = \int dV = \int_a^b 2\pi x\,(B - f(x))\,dx$$

$$\Delta V = 2\pi x_i \cdot \Delta x\,(B - f(x_i))$$

$$dV = 2\pi x\,(B - f(x))\,dx$$

Hvis $f(x)$ er integrerbar, så er også $2\pi x\,(B - f(x))$ det.

eksempel



Vil finne volumet til kula minus hullet!

$$\underline{V = \int dV}$$

$f(x) = \sqrt{1-x^2} \qquad x \in [0,1]$

$$dV = 2f \cdot dx \cdot 2\pi x = 4\pi x\, f(x)\, dx = 4\pi x \sqrt{1-x^2}\, dx$$

$R = \Delta x \cdot 2 \cdot f$

$\Delta V = R \cdot 2\pi x_i$

volum til sylinderskall

(f er integrerbar, så $4\pi x f$ er også det).

$$\underline{V} = \int_{\frac{1}{2}}^{1} 4\pi x \sqrt{1-x^2}\, dx = 4\pi \int_{\frac{1}{2}}^{1} x\sqrt{1-x^2}\, dx = 4\pi \left[-\tfrac{1}{3}(1-x^2)^{\frac{3}{2}} \right]_{\frac{1}{2}}^{1}$$

$$= 0 - 4\pi\left(-\tfrac{1}{3}\left(1-\tfrac{1}{4}\right)^{\frac{3}{2}}\right)$$

$$= \tfrac{4\pi}{3}\left(\tfrac{3}{4}\right)^{3/2} = *$$

$$\int x\sqrt{1-x^2}\, dx = -\tfrac{1}{2}\int -2x\sqrt{1-x^2}\, dx = -\tfrac{1}{2}\int u^{\frac{1}{2}}\, du = -\tfrac{1}{2}\,\frac{1}{1+\frac{1}{2}}\, u^{1+\frac{1}{2}} + C$$

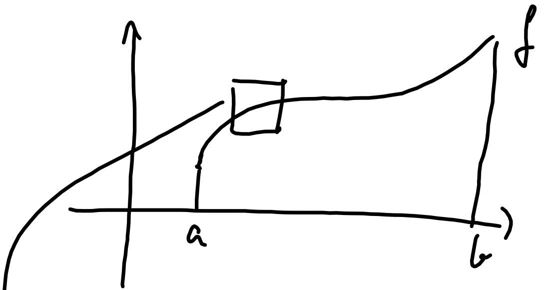
$u = 1 - x^2, \quad \frac{du}{dx} = -2x \Rightarrow du = -2x\, dx$

$$= -\tfrac{1}{2}\cdot\tfrac{2}{3}\, u^{3/2} + C$$

$$= \underline{\underline{-\tfrac{1}{3}(1-x^2)^{3/2} + C}}$$

$$* \quad \frac{4\pi}{3}\left(\frac{3}{4}\right)^{\frac{3}{2}} = \frac{4\pi}{3}\,\frac{3\cdot\sqrt{3}}{4\cdot\sqrt{4}} = \underline{\underline{\frac{\pi}{2}\sqrt{3}}}$$

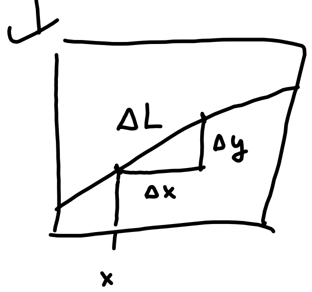
Lengdemåling:



Vil finne lengden L til grafen til f.

Anta at f er deriverbar og har kontinuerlig derivert.

$$L = \int dL$$



$$\Delta L = \sqrt{\Delta x^2 + \Delta y^2} = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\, \Delta x$$

$$\frac{\Delta y}{\Delta x} = \underline{\frac{f(x+\Delta x) - f(x)}{\Delta x}} \longrightarrow f'(x)$$

$$dL = \underbrace{\sqrt{1 + (f'(x))^2}}_{\text{integrerbart!}} \cdot dx$$

$$L = \int_a^b dL = \int_a^b \sqrt{1 + (f'(x))^2}\, dx \qquad !$$

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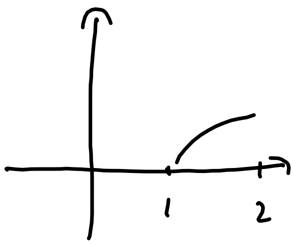
ex: $f(x) = \frac{2}{3}(x-1)^{\frac{3}{2}} \qquad x \in [1, 2]$



$$L = \int dL = \int_1^2 \sqrt{1 + (f'(x))^2}\, dx$$

$$f'(x) = \frac{2}{3} \cdot \frac{3}{2} \cdot (x-1)^{\frac{3}{2}-1} = (x-1)^{\frac{1}{2}}$$

$$\underline{\underline{L}} = \int_1^2 \sqrt{1 + ((x-1)^{\frac{1}{2}})^2}\, dx = \int_1^2 x^{\frac{1}{2}}\, dx = \left[\frac{1}{1+\frac{1}{2}} x^{1+\frac{1}{2}}\right]_1^2$$

$$= \frac{2}{3} 2^{\frac{3}{2}} - \frac{2}{3} \cdot 1^{\frac{3}{2}}$$

$$= \underline{\underline{\frac{2}{3}(2\sqrt{2} - 1)}}$$