FLVA
2.1-3
4-6 (neste 3 ganger)

Funksjoner i flere variable

$f : \mathbb{R} \to \mathbb{R} \qquad D_f = \mathbb{R}$

$f : A \to \mathbb{R} \qquad A \subseteq \mathbb{R}$

$A = D_f$ definisjonsområde

$$\vec{F} : \mathbb{R}^n \to \mathbb{R}^m$$

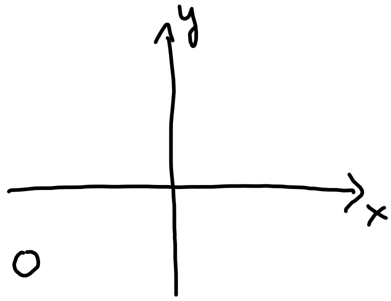
alternativt: $\vec{F} : A \to \mathbb{R}^m \qquad A \subseteq \mathbb{R}^n$

$A = D_{\vec{F}}$ definisjons-område

eks

$$\vec{F}(x,y) = \left(\frac{1}{x}, \frac{1}{y}, x \cdot y\right) \in \mathbb{R}^3$$

$$D_{\vec{F}} \stackrel{?}{=} \{(x,y) : xy \neq 0\} \subseteq \mathbb{R}^2$$



Hvordan ser grafen til $\vec{F}$ ut? Vet ikke

$x \gtrless 0$

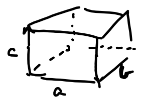
$y \gtrless 0$

$\vec{F} : A \to \mathbb{R}^3$, $A \subseteq \mathbb{R}^2$

Grafen til $\vec{F}$

$$= \{(x, y, F_1, F_2, F_3) : (x,y) \in A\} \subseteq \mathbb{R}^5$$

$$(x, y, \tfrac{1}{x}, \tfrac{1}{y}, xy)$$



$\vec{F} : \mathbb{R}^3 \to \mathbb{R}^2$

$\vec{F}(a,b,c) = (\underbrace{abc}_{\text{volum}}, \underbrace{2ab+2ac+2bc}_{\text{overflateareal}}) \leftarrow$

$A = \{(a,b,c) : a>0, b>0, c>0\} \subseteq \mathbb{R}^3$

$\vec{F} : \mathbb{R}^n \to \mathbb{R}^m$ kalles ofte en vektorfunksjon når $m>1$.

kalles en skalarfunksjon når $m=1$.

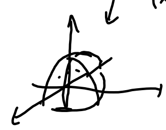
Hvis $n=2$ og $m=1$
blir grafen en flate i $\mathbb{R}^3$ (kan tegnes)

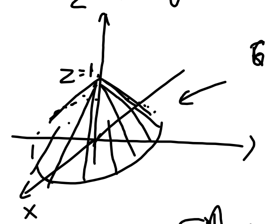
$F(x,y) = 1-x^2-y^2 \qquad A=\{(x,y) \mid x^2+y^2 \leq 1\}$

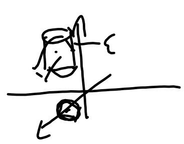
$F: A \to \mathbb{R} \qquad z = F(x,y)$

$G(x,y) = 1-\sqrt{x^2+y^2}$

Grafen $= \{(x,y,z) : z = 1-x^2-y^2,\ (x,y) \in A\}$



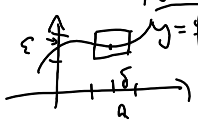




Kontinuitet:

$f : A \to \mathbb{R} \qquad a \in A \subseteq \mathbb{R}.$

f er kontinuerlig i a dersom det for hver $\varepsilon>0$ fins en $\delta>0$ slik at
$|f(x)-f(a)| < \varepsilon$ når $x \in A$ og $|x-a|<\delta$



$\vec{F} : A \to \mathbb{R}^m \qquad \vec{a} \in A \subseteq \mathbb{R}^n$

Def: $\vec{F}$ er kontinuerlig i $\vec{a}$ dersom det for hver $\varepsilon>0$ fins $\delta>0$ slik at
$|\vec{F}(\vec{x}) - \vec{F}(\vec{a})| < \varepsilon$ når $\vec{x} \in A$ og $|\vec{x}-\vec{a}| < \delta$.

Regneregler:

$\vec{F}, \vec{G} : \mathbb{R}^n \to \mathbb{R}^m$ er kontinuerlige i $\vec{a}$

så er $\vec{F}+\vec{G}$ kontinuerlig i $\vec{a}$
$\vec{F}\cdot\vec{G}$ kontinuerlig i $\vec{a}$

$\vec{F}+\vec{G} : \mathbb{R}^n \to \mathbb{R}^m$

$\vec{F}\cdot\vec{G} : \mathbb{R}^n \to \mathbb{R}$

Vil vise at $\vec{F}\cdot\vec{G}$ er kontinuerlig i $\vec{a}$ dersom $\vec{F}$ og $\vec{G}$ er det.

Bevis Gitt $\varepsilon > 0$ vil finne $\delta > 0$ s.a

$$\left|\vec{F}(\vec{x})\cdot\vec{G}(\vec{x}) - \vec{F}(\vec{a})\cdot\vec{G}(\vec{a})\right| < \varepsilon \quad \text{når } \underline{|\vec{x}-\vec{a}| < \delta}$$

undersøke

$$\left|\vec{F}(\vec{x})\cdot\vec{G}(\vec{x}) - \vec{F}(\vec{a})\cdot\vec{G}(\vec{a})\right| = \left|\underbrace{\vec{F}(\vec{x})\cdot\vec{G}(\vec{x}) - \vec{F}(\vec{a})\cdot\vec{G}(\vec{x})} + \underbrace{\vec{F}(\vec{a})\cdot\vec{G}(\vec{x}) - \vec{F}(\vec{a})\cdot\vec{G}(\vec{a})}\right|$$

$$\leq \left|\vec{F}(\vec{x})\cdot\underline{\vec{G}(\vec{x})} - \vec{F}(\vec{a})\cdot\underline{\vec{G}(\vec{x})}\right| + \left|\underline{\vec{F}(\vec{a})}\cdot\vec{G}(\vec{x}) - \underline{\vec{F}(\vec{a})}\cdot\vec{G}(\vec{a})\right|$$

$$= \left|(\vec{F}(\vec{x}) - \vec{F}(\vec{a}))\cdot\vec{G}(\vec{x})\right| + \left|\vec{F}(\vec{a})\cdot(\vec{G}(\vec{x}) - \vec{G}(\vec{a}))\right|$$

$|\vec{a}\cdot\vec{G}| \leq |\vec{a}|\cdot|\vec{G}|$

$$\leq \left|\underbrace{\vec{F}(\vec{x}) - \vec{F}(\vec{a})}_{? < \frac{\varepsilon}{2}}\right| \left|\underbrace{\vec{G}(\vec{x})}\right| + \left|\underbrace{\vec{F}(\vec{a})}\right| \left|\underbrace{\vec{G}(\vec{x}) - \vec{G}(\vec{a})}_{? < \frac{\varepsilon}{2}}\right|$$

$\Uparrow$

hvis $\quad |\vec{G}(\vec{x}) - \vec{G}(\vec{a})| < \dfrac{\varepsilon}{2|\vec{F}(\vec{a})|}$

$\Uparrow$

det fins en δ_1 s.a. $|\vec{x}-\vec{a}| \underline{< \delta_1}$

det fins en $\underline{\delta_2}$ s.a $|\vec{x}-\vec{a}| < \underline{\delta_2} \Rightarrow |\vec{G}(\vec{x}) - \vec{G}(\vec{a})| < \underline{1}$

$\Downarrow$

$|\vec{G}(\vec{x})| < \underline{|\vec{G}(\vec{a})| + 1}$

så kan vi finne δ_3 s.a

$|\vec{x}-\vec{a}| < \delta_3 \Rightarrow |\vec{F}(\vec{x}) - \vec{F}(\vec{a})| < \dfrac{\varepsilon}{2(|\vec{G}(\vec{a})|+1)}$

oppsummerer: velg $\underline{\delta} = \min(\delta_1, \delta_2, \delta_3)$

Da vil $|\vec{x}-\vec{a}| < \delta \Rightarrow$

$$\underline{\left|\vec{F}(\vec{x})\cdot\vec{G}(\vec{x}) - \vec{F}(\vec{a})\cdot\vec{G}(\vec{a})\right|} \leq \left|\underline{\vec{F}(\vec{x}) - \vec{F}(\vec{a})}\right|\cdot\left|\underline{\vec{G}(\vec{x})}\right| + \left|\vec{F}(\vec{a})\right|\left|\vec{G}(\vec{x}) - \vec{G}(\vec{a})\right|$$

$$< \frac{\varepsilon}{2(|\vec{G}(\vec{a})|+1)}(|\vec{G}(\vec{a})|+1) + |\vec{F}(\vec{a})|\frac{\varepsilon}{2|\vec{F}(\vec{a})|}$$

$$= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \underline{\varepsilon}.$$

<u>Setning</u>: $\vec{F} : \mathbb{R}^n \to \mathbb{R}^m$

$$\vec{F}(x_1, x_2, \dots, x_n) = (F_1(x_1 \dots x_n), F_2(x_1 \dots x_n) \dots, F_m(x_1 \dots x_n))$$

$\vec{F}$ er kontinuerlig i $\vec{a}$ hvis og bare hvis hver av de m skalarfunksjonene $F_1, \dots, F_m$ er kontinuerlige i $\vec{a}$.

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i $\mathbb{R}$ $\quad |x - a| < \delta \iff x \in (a-\delta, a+\delta)$

i $\mathbb{R}^n$ $\quad |\vec{x} - \vec{a}| < \delta \iff \vec{x} \in B(\vec{a}, \delta)$

$$|\vec{F}(\vec{x}) - \vec{F}(\vec{a})| < \varepsilon \quad \text{når } \vec{x} \in B(\vec{a}, \delta)$$

$$\Downarrow$$

$$|F_i(\vec{x}) - F_i(\vec{a})| < \varepsilon \qquad x_i \in (a_i - \delta, a_i + \delta)$$

$< \frac{\varepsilon}{m}$ for hver i $\quad \square$

<u>Grenseverdier</u>

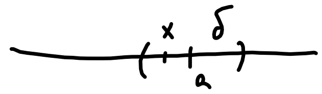
$$\lim_{\vec{x}\to\vec{a}} \vec{F}(\vec{x}) \overset{?}{=} \vec{L}$$

hvis det for hver $\varepsilon > 0$ finnes en $\delta > 0$ s.a. $|\underline{\underline{\vec{F}(\vec{x})}} - \vec{L}| < \varepsilon$ når $\vec{x} \in \underline{A}$ og $0 < |\vec{x} - \vec{a}| < \delta$ *

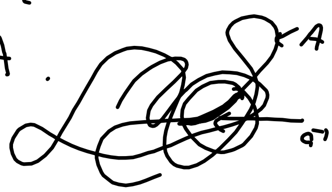
$A \subseteq D_{\vec{F}}$

$$\lim_{x\to a} f(x) = L$$

betyr at for hver $\varepsilon > 0$ finnes det en $\delta > 0$ s.a. for hver x, $0 < |x - a| < \delta$ så er $|f(x) - L| < \varepsilon$.



a er et <u>opphopnings punkt</u> for A.
<u>det vil si</u> at hver ball med sentrum i $\vec{a}$ inneholder uendelig mange punkter i fra A.



<u>eks:</u>

$$\vec{F}(x,y) = \frac{x^4 - y^4}{x^2 - y^2} = \frac{(x^2+y^2)(x^2-y^2)}{(x^2-y^2)} \qquad D_{\vec{F}} = \{(x,y) \mid x^2 - y^2 \neq 0\}$$

$$\lim_{(x,y)\to(0,0)} \vec{F}(x,y) = \lim_{(x,y)\to(0,0)} x^2 + y^2 = \underline{\underline{0}}$$