Vektorfunksjoner

$A \subseteq \mathbb{R}^n$: $f : A \longrightarrow \mathbb{R}^m$

$A = D_f$

Eksempel 1 a) $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$$f(x,y) = (xy,\ 1-\sin(x),\ y^2 \cdot x)$$

b) A $m\times n$-matrise

$\rightsquigarrow$ $T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ $\quad \vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$

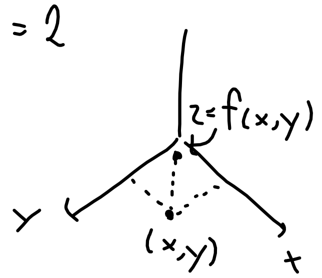
$T_A \vec{v} = A\vec{v}$ $\quad A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$

$$\underset{(m\times n)(n\times 1)}{A\vec{v}} = \underset{m\times 1}{\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}}$$

Skalarfelt

$f : A \longrightarrow \mathbb{R}$, $A \subseteq \mathbb{R}^n$

$n = 2$



Eksempel 2

a) $f(x,y) = \sqrt{1-x^2-y^2}$

Definisjonsmengden:

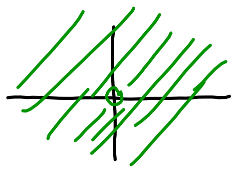
$1 - x^2 - y^2 \geq 0 \rightsquigarrow x^2+y^2 \leq 1$

b) $f(x,y) = \ln(x^2+y^2)$

Definisjonsmengden: Må ha

$x^2+y^2 > 0 \rightsquigarrow$ enten $x \neq 0$ eller $y \neq 0$

$$D_f = \{(x,y)\in\mathbb{R}^2 : x\neq 0 \text{ eller } y \neq 0\} = \mathbb{R}^2 \setminus \{(0,0)\}.$$

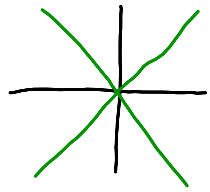


c) $f(x,y,z) = \dfrac{1}{x^2-y^2} + z$

D_f: Må ha $x^2-y^2=(x+y)(x-y) \neq 0$,
altså $x \neq -y$ og $x \neq y$.

$D_f = \{(x,y,z) \in \mathbb{R}^3 : x \neq y \text{ og } x \neq -y\}$

(Disse linjene er ikke med)



$\boxed{\text{Partiellderiverte}}$

$f: A \longrightarrow \mathbb{R}$ funksjon (skalarfelt),
$A \subseteq \mathbb{R}^n$, $\vec{a} \in A$, $1 \leq i \leq n$.
$\vec{a} = (a_1,\dots,a_n)$

$$\frac{\partial f}{\partial x_i}(a_1,\dots,a_n) = \lim_{h\to 0} \frac{f(a_1,\dots,a_i+h,\dots,a_n) - f(a_1,\dots,a_n)}{h}$$

Partiellderivert til f. $\dfrac{\partial f}{\partial x_1}, \dfrac{\partial f}{\partial x_2}, \dots, \dfrac{\partial f}{\partial x_n}$.

Eksempel 3 a) $f(x,y) = 3x^6 + 5xy^2 - y + 1$

$\dfrac{\partial f}{\partial x}(x,y) = 3\cdot 6x^5 + 5y^2 = 18x^5 + 5y^2$.

$\dfrac{\partial f}{\partial y}(x,y) = 0 + 5x(2y) - 1 = 10xy - 1$.

b) $f(x,y) = \ln(x^2+y^2)$.

$\dfrac{\partial f}{\partial x}(x,y) = \dfrac{1}{x^2+y^2}(2x+0) = \dfrac{2x}{x^2+y^2}$

$\dfrac{\partial f}{\partial y}(x,y) = \dfrac{1}{x^2+y^2}(0+2y) = \dfrac{2y}{x^2+y^2}$

c) $f(x,y,z) = \dfrac{1}{x^2-y^2} + z$

$\dfrac{\partial f}{\partial x}(x,y,z) = -\dfrac{1}{(x^2-y^2)^2}\cdot(2x) = -\dfrac{2x}{(x^2-y^2)^2}$

$\dfrac{\partial f}{\partial y}(x,y,z) = -\dfrac{1}{(x^2-y^2)^2}\cdot(-2y) = \dfrac{2y}{(x^2-y^2)^2}$

$\dfrac{\partial f}{\partial z}(x,y,z) = 1$

Gradienten til $f: A \longrightarrow \mathbb{R}$, $A \subseteq \mathbb{R}^n$, er en funksjon

$$\nabla f : A \longrightarrow \mathbb{R}^n$$

$$\nabla f(\vec{a}) = \left(\frac{\partial f}{\partial x_1}(\vec{a}), \frac{\partial f}{\partial x_2}(\vec{a}), \ldots, \frac{\partial f}{\partial x_n}(\vec{a}) \right)$$

Eksempel: $f(x,y,z) = \frac{1}{x^2 - y^2} + z$.

$$\nabla f(x,y,z) = \left(\frac{-2x}{(x^2-y^2)^2}, \frac{2y}{(x^2-y^2)^2}, 1 \right)$$

Deriverbarhet

Én dimensjon: $f: \mathbb{R} \to \mathbb{R}$

$$f(x+h) - f(x) \approx h f'(x) \quad \text{for } h \text{ liten.}$$

To variable: $f: \mathbb{R}^2 \to \mathbb{R}$

$$f(x+h, y+k) - f(x,y) \approx h \cdot \frac{\partial f}{\partial x}(x,y) + k \cdot \frac{\partial f}{\partial y}(x,y)$$
$$= (h,k) \cdot \nabla f(x,y)$$

n variable:

$$f(\vec{x} + \vec{r}) - f(\vec{x}) \approx \vec{r} \cdot \nabla f(\vec{x}) \quad \text{for } \vec{r} \text{ liten,}$$

dvs. $|\vec{r}|$ liten.

Definisjon $f: A \longrightarrow \mathbb{R}$, $A \subseteq \mathbb{R}^n$, er deriverbar i $\vec{a} \in A$ dersom alle partiellderiverte i $\vec{a}$ eksisterer, og

$$\lim_{\vec{r} \to 0} \frac{f(\vec{a} + \vec{r}) - f(\vec{a}) - \vec{r} \cdot \nabla f(\vec{a})}{|\vec{r}|} = 0.$$

Setning Hvis de partiellderiverte eksisterer og er kontinuerlige i en omegn rundt $\vec{a}$, så er f deriverbar i $\vec{a}$.

Retningsderivert

$f : A \longrightarrow \mathbb{R}$. $\vec{a} \in A$ $\vec{r} \in \mathbb{R}^n$
$A \subseteq \mathbb{R}^n$

$$f'(\vec{a};\vec{r}) = \lim_{h\to 0} \frac{f(\vec{a}+h\vec{r}) - f(\vec{a})}{h}$$

retningsderiverte i punktet $\vec{a}$ i retning $\vec{r}$.

Setning

Hvis f er deriverbar i $\vec{a}$, så er

$$f'(\vec{a};\vec{r}) = \nabla f(\vec{a}) \cdot \vec{r}$$

Eksempel 4 Regn ut den retningsderiverte til $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$, $f(x,y) = x^2 - y^2$ i punktet $(0,0)$ i retningen $\vec{r} \in \mathbb{R}^2$.

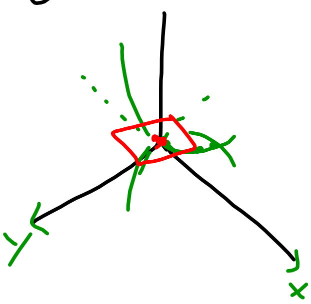
$$\frac{\partial f}{\partial x}(x,y) = 2x \qquad \frac{\partial f}{\partial y}(x,y) = -2y$$

$$\nabla f(x,y) = (2x, -2y). \qquad \vec{r} = (r_1, r_2).$$

$$\nabla f(0,0) = (2\cdot 0, -2\cdot 0) = (0,0)$$

$$f'(0,0;\vec{r}) = \nabla f(0,0) \cdot (r_1, r_2) = 0$$

Setning Anta at f er deriverbar i $\vec{a}$. Da peker $\nabla f(\vec{a})$ i retningen hvor f vokser hurtigst i $\vec{a}$, og stigningstallet til f i denne retningen er $|\nabla f(\vec{a})|$.



Bevis: $f'(\vec{a};\vec{r}) = \nabla f(\vec{a}) \cdot \vec{r} = |\nabla f(\vec{a})||\vec{r}| \cos\theta$

Dette er størst når $\cos\theta = 1$, dvs $\theta = 0$, altså når $\nabla f(\vec{a})$ og $\vec{r}$ er parallelle.

Eksempel 5 La $f(x,y) = x^2 y$, $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$. $\vec{a} = (3,2)$. Finn retningen f vokser raskest i, i punktet $\vec{a}$, og regn ut den retningsdoriverte i denne retningen.

Regner ut partiellderiverte:

$$\frac{\partial f}{\partial x}(x,y) = 2xy \qquad \frac{\partial f}{\partial y}(x,y) = x^2$$

$$\nabla f(x,y) = (2xy, x^2) \qquad \nabla f(3,2) = (2 \cdot 3 \cdot 2, 3^2) = (12, 9).$$

f vokser raskest i retning $\nabla f(3,2) = (12,9)$.

$$|\nabla f(3,2)| = \sqrt{12^2 + 9^2} = 15$$

$$\vec{u} = \frac{1}{15}(12,9) = \left(\frac{4}{5}, \frac{3}{5}\right).$$

Generelt: $\vec{v} \in \mathbb{R}^n$

$$\vec{u} := \frac{1}{|\vec{v}|}\vec{v}$$

$$|\vec{u}| = \left|\frac{1}{|\vec{v}|}\vec{v}\right| = \frac{1}{|\vec{v}|}|\vec{v}| = 1$$

Stigningstallet til f i denne retningen er gitt ved $|\nabla f(3,2)| = 15$.

Høyere ordens deriverte

$f: A \longrightarrow \mathbb{R}$, $A \subseteq \mathbb{R}$ $\rightsquigarrow$ f' første ordens derivert

$\rightsquigarrow$ $f'' = (f')'$ andre ordens derivert

$\rightsquigarrow \cdots\cdots$ $f^{(n)}$ n'te ordens derivert.

$f: A \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$, eks. $f(x,y)$.

$\frac{\partial f}{\partial x}$ partiell derivert mhp. x. $\frac{\partial f}{\partial y}$ førsteordens deriverte.

Fire ulike andre ordens deriverte:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial f}{\partial x} \qquad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\frac{\partial f}{\partial x}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\frac{\partial f}{\partial y} \qquad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\frac{\partial f}{\partial y}$$

Eks 6

$$f(x,y) = x^3y^2.$$

$$\frac{\partial f}{\partial x}(x,y) = 3x^2y^2 \qquad \frac{\partial f}{\partial y}(x,y) = 2x^3y$$

$$\frac{\partial^2 f}{\partial x^2}(x,y) = 3(2xy^2) = 6xy^2 \qquad \frac{\partial^2 f}{\partial y \partial x}(x,y) = 3x^2(2y)$$

$$= 6x^2y$$

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = 2(3x^2)y = 6x^2y$$

$$\frac{\partial^2 f}{\partial y^2}(x,y) = 2x^3$$