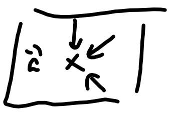
2.5 Høyere ordens partiellderiverte

Begynner repetisjon av forrige uke (2.3 - 2.4).

Skalarfelt: $f : \mathbb{R}^n \to \mathbb{R}$

$f(x,y) : \mathbb{R}^2 \to \mathbb{R}$

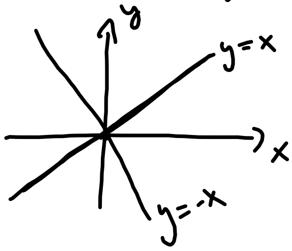
 $\subset \mathbb{R}^2$

$$f(x,y) = x \cdot y$$

$\frac{\partial f}{\partial x}(x,y) = y$, $\frac{\partial f}{\partial y}(x,y) = x$

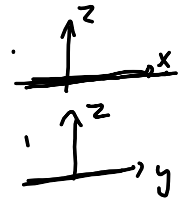
$\frac{\partial f}{\partial x}(1,1) = 1$ $\quad \frac{\partial f}{\partial y}(1,1) = 1$ $\quad \boxed{\nabla f(1,1) = (1,1)}$

i $(0,0)$: $\frac{\partial f}{\partial x}(0,0) = 0$ $\quad \frac{\partial f}{\partial y}(0,0) = 0$ $\quad \boxed{\nabla f(0,0) = (0,0)}$



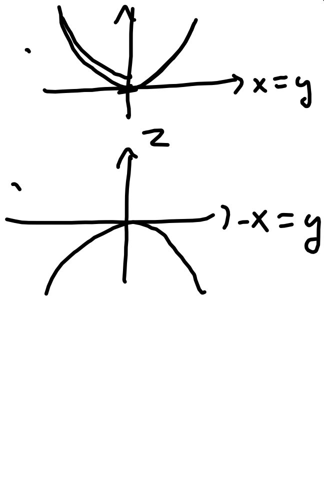
x-aksen : $y=0$: $f(x,0) = 0$

y-aksen : $x=0$: $f(0,y) = 0$



$y=x$: $f(x,x) = x^2$

$y=-x$: $f(x,-x) = -x^2$



$$f'(\vec{a}, \vec{r}) = \nabla f(\vec{a}) \cdot \vec{r}$$

$f: \mathbb{R}^2 \to \mathbb{R}$

$\nabla f(\vec{a}) = (0,0)$ for alle $\vec{a} \in \mathbb{R}^2$

hva er da f ?

f er konstant !

$f(x,y)$ Anta $\frac{\partial f}{\partial x}(x,y)$, $\frac{\partial f}{\partial y}(x,y)$ eksisterer for alle $\vec{a} \in \mathbb{R}^2$

$$\frac{\partial f}{\partial x}(x,y) : \mathbb{R}^2 \to \mathbb{R}$$

$$\frac{\partial f}{\partial y}(x,y) : \mathbb{R}^2 \to \mathbb{R}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}(x,y)\right) = \frac{\partial^2}{(\partial x)^2} f(x,y) \quad \frac{\partial}{\partial y}\frac{\partial f}{\partial x}(x,y) = \frac{\partial^2}{\partial y \partial x} f(x,y)$$

Anta at

$$\frac{\partial^2}{(\partial x)^2} f(x,y) = \frac{\partial^2}{\partial x \partial y} f(x,y) = \frac{\partial^2}{\partial y \partial x} f(x,y) = \frac{\partial^2}{(\partial y)^2} f(x,y) = 0$$

Hva er da f ?

$$\left[\begin{array}{l} f(x,y) = xy \Rightarrow \underline{\frac{\partial f}{\partial x}(x,y) = y} \quad \underline{\frac{\partial f}{\partial y}(x,y) = x} \\ \Rightarrow \frac{\partial^2 f}{\partial x^2}(x,y) = 0 \quad \underline{\frac{\partial^2 f}{\partial x \partial y}(x,y) = 1} \quad \underline{\frac{\partial^2 f}{\partial y \partial x}(x,y) = 1} \quad \frac{\partial^2 f}{\partial y^2} = 0 \end{array}\right]$$

$\underline{f(x,y) = x}$

$\underline{f(x,y) = y}$

$$f(x,y) = \underline{Ax + By + C}$$
(x^n)

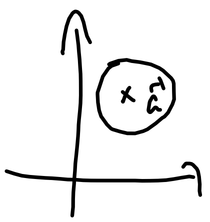
Anta

$$\frac{\partial^3}{(\partial x)^3} f(x,y) = \frac{\partial^3}{(\partial x)^2 \partial y} f(x,y) = \frac{\partial^3}{\partial x (\partial y)^2} f(x,y) = \frac{\partial^3}{(\partial y)^3} f(x,y) = 0$$

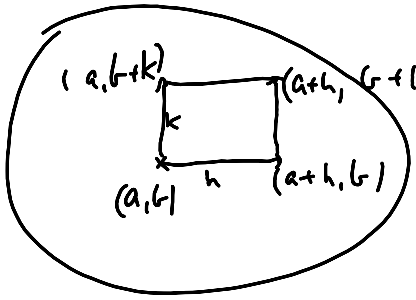
Hva er $f(x,y)$?

$$f(x,y) = Ax^2 + Bx + Cy^2 + Dy + E + Fxy$$

Setning: Dersom $f : \mathbb{R}^2 \to \mathbb{R}$ er et skalarfelt slik at $\frac{\partial^2}{\partial x \partial y} f$ og $\frac{\partial^2}{\partial y \partial x} f$ begge er definert i $\vec{a} \in \mathbb{R}^2$ og er kontinuerlige i en sirkelomegn om $\vec{a}$, da er $\frac{\partial^2}{\partial x \partial y} f(\vec{a}) = \frac{\partial^2}{\partial y \partial x} f(\vec{a})$.



$\vec{a} = (a, b)$



f deriverbar i $\vec{a}$:

$$\lim_{\vec{r} \to \vec{0}} \frac{f(\vec{a} + \vec{r}) - f(\vec{a}) - \overbrace{\nabla f(\vec{a}) \cdot \vec{r}}^{f'(\vec{a}, \vec{r})}}{|\vec{r}|} = 0$$

$$\Delta(h,k) = f(a+h, b+k) - f(a, b+k) - f(a+h, b) + f(a,b)$$

Vil vise at $\frac{\Delta(h,k)}{h \cdot k} \approx \frac{\partial^2}{\partial x \partial y} f(a,b) = \frac{\partial^2}{\partial y \partial x} f(a,b)$ når $(h,k) \to (0,0)$.

$$f(x,y) \qquad (a,b) \qquad \big(g(y) = f(a+h,y) - f(a,y)\big)$$

$$g(x) = f(x, b+k) - f(x,b) \qquad c \in (a, a+h)$$

$$\underline{g(a+h) - g(a)} = \underline{g'(c)\cdot h}$$

$$= f(a+h, b+k) - f(a+h, b) - f(a, b+k) + f(a,b) = \Delta(h,k)$$

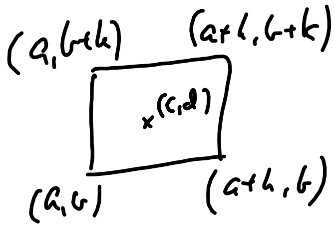
$$g'(c)\cdot h = \Delta(h,k)$$

$$\big(f'(c, b+k) - f'(c,b)\big)\cdot h$$

$$\Big(\underbrace{\frac{\partial f}{\partial x}(c, \underline{b+k}) - \frac{\partial f}{\partial x}(c, \underline{b})}\Big)\cdot h$$

$$= \frac{\partial}{\partial y}\Big(\frac{\partial f}{\partial x}\Big)(c,d)\cdot k \cdot h \qquad d \in (b, b+k)$$

$$\underline{\Delta(h,k)} = \frac{\partial^2}{\partial y \partial x}\underline{f(c,d)} \cdot \underline{k\cdot h} \rightarrow$$



$$\frac{\Delta(h,k)}{k\cdot h} = \underline{\underline{\frac{\partial^2}{\partial y \partial x} f(c,d)}}$$

$c \in (a, a+h)$
$d \in (b, b+k)$

siden $\frac{\partial^2}{\partial y \partial x} f$ er kontinuerlig i (a,b), så er $\underline{\frac{\partial^2}{\partial y \partial x} f(a,b)} = \underline{\underline{\lim_{k,h \to (0,0)} \frac{\Delta(h,k)}{h\cdot k}}}$

$$= \underline{\frac{\partial^2}{\partial x \partial y} f(a,b)}$$

$$f(x,y) = \begin{cases} \dfrac{x^3y - xy^3}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases}$$

Da er $\frac{\partial^2}{\partial x \partial y} f$ og $\frac{\partial^2}{\partial y \partial x} f$ begge definert i $(0,0)$, men de er forskjellige !!

La $\vec{F}(x,y,z) : \mathbb{R}^3 \to \mathbb{R}^2$

$$\vec{F}(x,y,z) = \left(F_1(x,y,z), F_2(x,y,z)\right)$$

Hva er den deriverte til $\vec{F}$?

Jacobimatrisen til $\vec{F}$. $=$
$$\begin{pmatrix} \frac{\partial F_1}{\partial x}(x,y,z) & \frac{\partial F_1}{\partial y}(x,y,z) & \frac{\partial F_1}{\partial z}(x,y,z) \\ \frac{\partial F_2}{\partial x}(x,y,z) & \frac{\partial F_2}{\partial y}(x,y,z) & \frac{\partial F_2}{\partial z}(x,y,z) \end{pmatrix}$$