Integrasjonsteknikker.

- delvis integrasjon
- substitusjon.

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u, v deriverbare fn

$$(u \cdot v)' = u'v + uv'$$

$$\int (u \cdot v)' \, dx = \int u' \cdot v \, dx + \int u v' \, dx$$

$$u \cdot v + C = \int u' v \, dx + \int u v' \, dx$$

ubestemte integraler

$$\Rightarrow \quad uv = \int u' v \, dx + \int u v' \, dx$$

$$\boxed{\int \underline{u' v} \, dx = u \cdot v - \int u v' \, dx}$$

ex:

$$\int \underline{x \, e^x} \, dx =$$

Hva skal jeg velge for u' og hva skal jeg velge for v?

$$u' = e^x \qquad v = x$$
$$u = e^x \qquad v' = \underline{1}$$

$$u' = x \qquad v = e^x$$
$$u = \tfrac{1}{2}x^2 \qquad v = e^x$$
$$= \tfrac{1}{2}x^2 e^x - \underline{\int \tfrac{1}{2} x^2 e^x \, dx}$$

$$\underline{\int x e^x dx} = x e^x - \int e^x \, dx$$
$$= \underline{x e^x - e^x + C}$$

ex:

$$\int \sin^2 x \, dx$$

$$\sin^2 x = \sin x \cdot \sin x = \underline{1} \cdot \underline{\sin^2 x}$$

$$u' = \sin x \qquad v = \sin x$$
$$u = -\cos x \qquad v' = \cos x$$

$$\cos^2 x + \sin^2 x = 1$$
$$\cos^2 x = 1 - \sin^2 x$$

$$\underline{\underline{\int \sin^2 x \, dx}} = -\sin x \cos x - \int (-\cos x) \cos x \, dx$$
$$= -\sin x \cos x + \int \cos^2 x \, dx$$
$$= -\sin x \cos x + \int 1 - \sin^2 x \, dx$$
$$= -\sin x \cos x + \int dx - \underline{\int \sin^2 x \, dx}$$

$$\underline{2 \int \sin^2 x \, dx = -\sin x \cos x + \int dx = -\sin x \cos x + x + C'}$$

$$\underline{\int \sin^2 x \, dx} = \underline{-\tfrac{1}{2} \sin x \cos x + \tfrac{1}{2} x + C}$$

ex $\int \underline{1 \cdot \ln x} \, dx = x \ln x - \int x \cdot \frac{1}{x} \, dx$

$u' = 1 \quad v = \ln x$

$u = x \quad v' = \frac{1}{x}$

$= x \ln x - \int dx$

$= x \ln x - x + C$

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ex $\int x \cdot \arctan x \, dx$

$u' = x \quad v = \arctan x$

$u = \frac{1}{2}x^2 \quad v' = \frac{1}{1+x^2}$

$= \frac{1}{2}x^2 \cdot \arctan x - \int \frac{\frac{1}{2}x^2}{1+x^2} \, dx$

$= \frac{1}{2}x^2 \arctan x - \frac{1}{2} \int \frac{1+x^2-1}{1+x^2} \, dx$

$= \frac{1}{2}x^2 \arctan x - \frac{1}{2} \int 1 - \frac{1}{1+x^2} \, dx$

$= \frac{1}{2}x^2 \arctan x - \frac{1}{2} \int dx + \frac{1}{2} \underbrace{\int \frac{1}{1+x^2} dx}_{= \frac{1}{2} \arctan x + C}$

$= \underline{\frac{1}{2}x^2 \arctan x - \frac{1}{2}x + \frac{1}{2} \arctan x + C}$

ex

$$\underline{I_n} = \int x^n e^x\,dx = x^n e^x - \int n x^{n-1} e^x\,dx$$

$u' = e^x \qquad v = x^n$

$u = e^x \qquad v' = n x^{n-1}$

$$= x^n e^x - n \underbrace{\int x^{n-1} e^x\,dx}$$

$$I_{n-1} = \int x^{n-1} e^x\,dx$$

$$= x^n e^x - n\, I_{n-1}$$

$$\underline{I_n} = \underbrace{x^n e^x - n I_{n-1}}$$

$$= x^n e^x - n\left(\underbrace{x^{n-1} e^x - (n-1) I_{n-2}}\right)$$

$$I_0 = \int x^0 e^x\,dx = \int e^x\,dx = e^x + C$$

$$= x^n e^x - n x^{n-1} e^x + n(n-1) I_{n-2}$$

$$=$$

$$= x^n e^x - n x^{n-1} e^x + n(n-1) x^{n-2} e^x - \dots + (-1)^n n(n-1)\cdots 2\cdot 1\cdot I_0$$

$$= x^n e^x - n x^{n-1} e^x + \dots + (-1)^n n(n-1)\cdot\ldots\cdot 1\cdot e_x + C$$

$$= e^x \sum_{k=1}^{n} (-1)^k n(n-1)\ldots(n-k+1)\, x^{n-k} + x^n e^x + C$$

$\boxed{0! = 1}$

$$= e^x \cdot \sum_{k=0}^{n} (-1)^k \frac{n!}{(n-k)!} x^{n-k} + C$$

Substitusjon.

La g være deriverbar, f kontinuerlig, F antiderivert til f $(F' = f)$

$$\int f(g(x))\,g'(x)\,dx = F(g(x)) + C$$

$$\underline{f(g(x))\,g'}(x) = F(g(x))' = F'(g(x))\,g'(x)$$
$$= \underline{f(g(x))\,g'(x)}$$

$u = g(x)$ (kjernen) $\underline{\frac{du}{dx} = g'(x)} \Rightarrow \underline{du = g'(x)\,dx}$

$F' = f$

$$\int f(g(x))\,\underbrace{g'(x)\,dx}_{du} = \int f(u)\,du$$
$$= F(u) + C$$
$$= \underline{F(g(x)) + C}$$

ex
$$\int \underline{\frac{1}{\sqrt{x}+1}\,dx}$$

Hva skal jeg velge som $u = g(x)$?

$\underline{u} = g(x) = \underline{\sqrt{x}+1} \Rightarrow \sqrt{x} = u - \underline{1}$

$u' = \frac{du}{dx} = \frac{1}{2}\frac{1}{\sqrt{x}} = \frac{1}{2}\frac{1}{\sqrt{x}}$ $\quad du = \frac{1}{2}\frac{1}{\sqrt{x}}\cdot dx$

$\boxed{\int \underline{u'\,dx} = u} = \int \underline{du}$ $\quad du = \frac{1}{2}\frac{1}{u-1}\,dx$

$\frac{du}{dx}\cancel{dx}$ $\quad \underline{2(u-1)\,du = dx}$

$$\int \frac{1}{\sqrt{x}+1}\,dx = \int \frac{1}{u}\,2(u-1)\,du = 2\int 1 - \frac{1}{u}\,du$$

$$= 2u - 2\ln|u| + C'$$

$u = \sqrt{x}+1$
$$= \underline{2\sqrt{x} + 2 - 2\ln(\sqrt{x}+1) + C'}$$
$$= 2\sqrt{x} - 2\ln(\sqrt{x}+1) + C$$

$$\int \cos\sqrt{x}\; dx = \int \cos u \; 2u\, du = 2\int u \cos u \; du$$

$u = g(x) = \sqrt{x}$

$\frac{du}{dx} = \frac{1}{2\sqrt{x}} = \frac{1}{2u} \Rightarrow dx = 2u\, du$

$v = u \quad w' = \cos u$

$v' = 1 \quad w = \sin u$

$$= 2u\sin u - 2\int \sin u \; du$$

$$= 2u \sin u + 2\cos u + C$$

$$= 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C$$

ex

$$\int \sin^2 x \; dx = \int \frac{1}{2} - \frac{1}{2}\cos 2x \; dx$$

$\cos 2x = \cos^2 x - \sin^2 x$

$= 1 - 2\sin^2 x$

$2\sin^2 x = 1 - \cos 2x$

$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$

$$= \int \frac{1}{2}\, dx - \frac{1}{2}\int \cos 2x \; dx$$

$$= \frac{1}{2}x - \frac{1}{2}\int \cos u \cdot \frac{1}{2} du$$

$u = 2x \quad du = 2\,dx \quad dx = \frac{1}{2}du$

$\frac{du}{dx} = 2$

$$= \frac{1}{2}x - \frac{1}{4}\int \cos u \, du$$

$$= \frac{1}{2}x - \frac{1}{4}\sin u + C$$

$$= \frac{1}{2}x - \frac{1}{4}\underbrace{\sin 2x}_{= 2\sin x \cos x} + C$$

$-\frac{1}{2}\sin x \cos x + \frac{1}{2}x + C$

$$\int f(g(x))\,dx \qquad \boxed{\int f(g(x))\,g'(x)\,dx \stackrel{?}{=}}$$

$u = g(x) \qquad x = h(u) \qquad h = g^{-1}$ (g strictly monotone).

$\frac{du}{dx} = g'(x) \qquad du = g'(x)\,dx$

$dx = \frac{du}{g'(x)} = \frac{1}{g'(x)}\,du = h'(u)\,du$

$\frac{dx}{du} = h'(u)$

$dx = h'(u)\,du$

$$\int f(g(x))\,dx = \int f(u)\,h'(u)\,du.$$

$$\int_0^{\frac{1}{2}} \frac{dx}{a^2\cos^2 x + b^2 \sin^2 x}$$

$u = \tan x \qquad \frac{du}{dx} = \frac{1}{\cos^2 x} = 1 + \tan^2 x = 1 + u^2$

$dx = \frac{du}{1+u^2}.$

$$\int_0^{\frac{1}{2}} \frac{\frac{1}{\cos^2 x} = 1+\tan^2 x}{a^2 + b^2 \frac{\sin^2 x}{\cos^2 x}}\,dx = \int_0^{\tan\frac{1}{2}} \frac{(1+u^2)\cdot \frac{du}{1+u^2}}{a^2 + b^2u^2} = \int_0^{\tan\frac{1}{2}} \frac{du}{a^2 + b^2u^2}$$

$x = 0 \qquad x = \frac{1}{2}$

$u = \tan 0 = 0 \qquad u = \tan\frac{1}{2}$

$$= \int_0^{\tan\frac{1}{2}} \frac{\frac{1}{a^2}\,du}{1 + \frac{b^2u^2}{a^2}} = \int_0^{\frac{b}{a}\tan\frac{1}{2}} \frac{\frac{1}{a^2}\cdot\frac{a}{b}\,dv}{1+v^2}$$

$v = \frac{bu}{a} \Rightarrow dv = \frac{b}{a}\,du \qquad du = \frac{a}{b}\,dv$

$$= \frac{1}{ab}\int_0^{\frac{b}{a}\tan\frac{1}{2}} \frac{dv}{1+v^2} = \frac{1}{ab}\Big[\arctan v\Big]_0^{\frac{b}{a}\tan\frac{1}{2}}$$

$$= \frac{1}{ab}\arctan\left(\frac{b}{a}\tan\frac{1}{2}\right) + C$$