9.3 Delbrøkoppspalting

Hva er

$$\int \frac{P(x)}{Q(x)}\,dx \quad \text{der}$$

$P(x)$ og $Q(x)$ er polynomer ?

I $\quad \int x^n\,dx = \frac{1}{n+1}x^{n+1} + C \quad$ n helt tall

forskjellig fra -1.

II $\quad \int \frac{1}{x}\,dx = \ln|x| + C$

III $\quad \int \frac{1}{1+x^2}\,dx = \arctan x + C$

IV $\quad \int \frac{1}{(1+x^2)^n}\,dx \overset{?}{=} \quad n > 1$ (helt tall)

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eksempel:

I

$$\int \frac{1}{(2x+1)^2}\,dx = \int \frac{1}{u^2}\cdot\frac{1}{2}du = \frac{1}{2}\int u^{-2}du$$

$u = 2x+1$

$du = 2\cdot dx \Rightarrow dx = \frac{1}{2}du$

$$= \frac{1}{2}\int u^{-2}du = \frac{1}{2(-2+1)}u^{-2+1} + C$$

$$= -\frac{1}{2}\,\frac{1}{u} + C = -\frac{1}{2}\,\frac{1}{2x+1} + C$$

$$\int \frac{1}{3+2x^2}\,dx \overset{\text{III}}{=} \int \frac{\frac{1}{3}}{1+\frac{2}{3}x^2}\,dx$$

$$\cancel{u = 3+2x^2}$$
$$\cancel{du = 4x\,dx}$$
$$\cancel{dx = \frac{du}{4x}}$$

$$= \int \frac{\frac{1}{3}}{1+\left(\sqrt{\frac{2}{3}}x\right)^2}\,dx$$

$$u = \sqrt{\frac{2}{3}}\,x \qquad du = \sqrt{\frac{2}{3}}\cdot dx$$
$$\Rightarrow dx = \sqrt{\frac{3}{2}}\,du$$

$$= \frac{1}{3}\int \frac{1}{1+u^2}\sqrt{\frac{3}{2}}\,du = \frac{1}{3}\cdot\sqrt{\frac{3}{2}}\int \frac{du}{1+u^2}$$

$$= \frac{1}{\sqrt{6}}\arctan u + C$$

$$= \frac{1}{\sqrt{6}}\arctan\left(\frac{\sqrt{2}}{\sqrt{3}}\cdot x\right) + C$$

$$\int \frac{2x+3}{x^2+2x+4}\,dx = \int \frac{2x+2 \quad +1}{x^2+2x+4}\,dx$$

$u = x^2+2x+4$

$du = (2x+2)\cdot dx$

$$= \int \frac{2x+2}{x^2+2x+4}\,dx + \int \frac{1}{x^2+2x+4}\,dx$$

$$= \int \frac{du}{u} + \int \frac{1}{x^2+2x+4}\,dx$$

$$= \ln|u| + \int \frac{1}{x^2+2x+1 \;+3}\,dx$$

$$= \ln|u| + \int \frac{1}{3+(x+1)^2}\,dx$$

$$= \ln|u| + \int \frac{\frac{1}{3}}{1+\frac{1}{3}(x+1)^2}\,dx$$

$$= \ln|u| + \frac{1}{3}\int \frac{1}{1+\left(\frac{x+1}{\sqrt{3}}\right)^2}\,dx$$

$v = \frac{x+1}{\sqrt{3}}$

$dv = \frac{1}{\sqrt{3}}\,dx \Rightarrow dx = \sqrt{3}\cdot dv$

$$= \ln|u| + \frac{1}{3}\int \frac{1}{1+v^2}\,\sqrt{3}\cdot dv$$

$$= \ln|u| + \frac{1}{\sqrt{3}}\arctan v + C$$

$$= \ln|x^2+2x+4| + \frac{1}{\sqrt{3}}\arctan\left(\frac{x+1}{\sqrt{3}}\right) + C$$

Hva med det generelle problemet:

$$\int \frac{P(x)}{Q(x)}\,dx \;?$$

graden til $P(x) \geq$ graden $Q(x)$?

Ja $\Rightarrow$ polynomdivisjon.

$$P(x) : Q(x) = S(x) + \frac{R(x)}{Q(x)}$$

der graden til $R(x) <$ graden til $Q(x)$.

Faktoriser Q:

$$Q(x) = \alpha\,(x-r_1)^{n_1}\cdots(x-r_i)^{n_i}\,(x^2-a_1x+b_1)^{m_1}\cdots(x^2-a_jx+b_j)^{m_j}$$

$$\frac{P(x)}{Q(x)} = \alpha\cdot\Big(\frac{A_{11}}{x-r_1} + \frac{A_{12}}{(x-r_1)^2} + \dots + \frac{A_{1n_1}}{(x-r_1)^{n_1}} +$$

$$+ \frac{A_{i1}}{x-r_i} + \dots + \frac{A_{in_i}}{(x-r_i)^{n_i}} + \frac{B_{11}x + C_{11}}{x^2-a_1x+b_1}$$

$$+ \dots + \frac{B_{1m_1}x + C_{1m_1}}{(x^2-a_1x+b_1)^{m_1}} + \_ + \_\Big).$$

eksempel

$$\int \frac{1}{x^2-1}\,dx =$$

① $x^2-1 = (x-1)(x+1)$

② $\frac{1}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1} = \frac{A(x+1)+B(x-1)}{(x-1)(x+1)} = \frac{(A+B)x + A-B}{x^2-1}$

$\Leftrightarrow \quad 1 = (A+B)x + (A-B)$

$\Leftrightarrow \quad A+B=0 \qquad A-B=1 \quad \Leftrightarrow \quad A=\frac{1}{2} \quad B=-\frac{1}{2}$.

③ $\int \frac{1}{x^2-1}\,dx = \int \left(\frac{\frac{1}{2}}{x-1} - \frac{\frac{1}{2}}{x+1}\right) dx$

$= \frac{1}{2}\int \frac{dx}{x-1} - \frac{1}{2}\int \frac{dx}{x+1} \overset{II}{=} \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C$

eksempel $\displaystyle\int \frac{2x^3-1}{(x-1)^2(x^2+1)}\,dx$

$$\frac{2x^3-1}{(x-1)^2(x^2+1)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1}$$

$$= \frac{A(x-1)(x^2+1) + B(x^2+1) + (Cx+D)(x-1)^2}{(x-1)^2(x^2+1)}$$

$$A(x-1)(x^2+1) = Ax^3 - Ax^2 + Ax - A$$

$$+ \quad B(x^2+1) = Bx^2 + B$$

$$+ \quad (Cx+D)(x^2-2x+1) = Cx^3 - (2C-D)x^2 - (2D-C)x + D$$

$$= (A+C)x^3 - (A-B+2C-D)x^2 + (A-2D+C)x - A+B+D$$

$$2x^3 - 1 = (A+C)x^3 - (A-B+2C-D)x^2 + (A-2D+C)x - A+B+D$$

$\Leftrightarrow$

$$\begin{aligned} A+C &= 2 \\ A-B+2C-D &= 0 \\ A+C-2D &= 0 \\ -A+B+D &= -1 \end{aligned}$$

$C = 2 - A$

$A - B + 4 - 2A - D = 0 \Rightarrow -A - B - D = -4$

$A + 2 - A - 2D = 0 \Rightarrow 2 - 2D = 0 \Rightarrow D = 1$

$-A + B + 1 = -1 \Rightarrow A = B + 2$

$-B - 2 - B - \underset{1}{D} = -4 \Rightarrow -2B = -4 + 3$

$B = \frac{1}{2}$

$D = 1 \quad B = \frac{1}{2} \quad A = B + 2 = \frac{5}{2} \quad C = 2 - A = -\frac{1}{2}$

$$\int \frac{2x^3-1}{(x-1)^2(x^2+1)}\,dx = \underbrace{\int \frac{\frac{5}{2}}{x-1}\,dx}_{II} + \underbrace{\int \frac{\frac{1}{2}}{(x-1)^2}\,dx}_{I} + \underbrace{\int \frac{-\frac{1}{2}x+1}{x^2+1}\,dx}_{I\ +\ III}$$

$v = x^2+1 \quad dv = 2x\,dx$

$$= \frac{5}{2}\int \frac{dx}{x-1} + \frac{1}{2}\int \frac{dx}{(x-1)^2} + \int \frac{-\frac{1}{2}\cdot 2x \cdot \frac{1}{2}}{x^2+1}\,dx - \frac{1}{2}\int \frac{dx}{x^2+1}$$

$u = x-1 \quad du = dx$

$$= \frac{5}{2}\int \frac{du}{u} + \frac{1}{2}\int \frac{du}{u^2} + \left(-\frac{1}{4}\right)\int \frac{dv}{v} - \frac{1}{2}\int \frac{dx}{1+x^2}$$

$$= \frac{5}{2}\ln|u| + \frac{1}{2}\left(-\frac{1}{u}\right) - \frac{1}{4}\ln|v| - \frac{1}{2}\arctan x + C$$

$$= \frac{5}{2}\ln|x-1| + \frac{1}{2}\,\frac{-1}{(x-1)} - \frac{1}{4}\ln(x^2+1) - \frac{1}{2}\arctan x + C.$$

IV $I_n = \int \frac{1}{(1+u^2)^n}\,du$ $I_{n-1} = \int \frac{1}{(1+u^2)^{n-1}}\,du$

$$\underline{I_n} = \frac{1}{2(n-1)} \frac{u}{(1+u^2)^{n-1}} + \frac{2n-3}{2(n-1)} \cdot \underline{I_{n-1}}$$