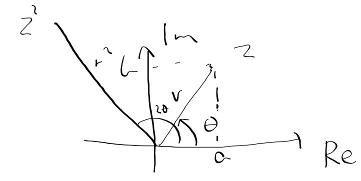
de Moivres formel

La z være et komplekst tall med polarkoordinater (r_1, θ_1)

La w ——— " ——— (r_2, θ_2)

Da er $z \cdot w$ et komplekst tall med polarkoordinater $(r_1 r_2, \theta_1 + \theta_2)$



$$z = a + ib = r_1 \cos\theta_1 + i r_1 \sin\theta_1 = r_1(\cos\theta_1 + i \sin\theta_1)$$

$$z \cdot z = z^2 = r_1 \cdot r_1 \cos(\theta_1 + \theta_1) + i r_1 r_1 \sin(\theta_1 + \theta_1)$$
$$= r_1^2(\cos 2\theta_1 + i \sin 2\theta_1)$$

potens $\quad z^n = r_1^n(\cos(n\theta_1) + i \sin n\theta_1)$

$z = a + ib \qquad$ hvis $b = 0$ så bør $e^z = e^a$

Def: $\quad e^z = e^a \cdot (\cos b + i \sin b)$

$$z = i\pi \Rightarrow e^z = e^{i\pi} = e^0(\cos\pi + i \sin\pi) = -1$$

$k = 0, \pm 1, \ldots \quad e^{i \cdot 2k\pi} = e^0(\cos 2k\pi + i \sin 2k\pi) = 1$

$$e^{z+w} \overset{?}{=} e^z \cdot e^w \quad \checkmark$$

$z = a + ib \qquad w = c + id$

$$e^{z+w} = e^{(a+c) + i(b+d)} = e^{a+c}(\cos(b+d) + i \sin(b+d))$$

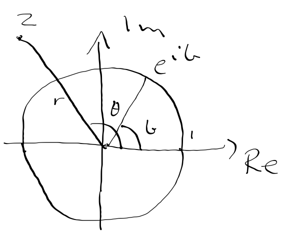
e^{z+w} har polarkoordinater $(e^{a+c}, b+d)$

$$e^z \cdot e^w = e^a(\cos b + i \sin b) \cdot e^c(\cos d + i \sin d)$$
$$= e^a \cdot e^c(\cos b + i \sin b)(\cos d + i \sin d)$$
$$= e^{a+c}(\cos b \cos d + \underset{=-1}{i^2} \sin b \sin d + i \cos b \sin d + i \sin b \cos d)$$
$$= e^{a+c}(\cos(b+d) + i \sin(b+d)) \leftarrow$$

$e^z \cdot e^w$ har polarkoordinater $(e^{a+c}, b+d)$

$a = 0 \qquad e^{ib} = \cos b + i \sin b$

$$|e^{ib}| = 1 = \sqrt{\cos^2 b + \sin^2 b}$$



$$z = r e^{i\theta} = r(\cos\theta + i \sin\theta)$$

$\cancel{a + ib = (r, \theta)}$

de Moivres formel

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$(e^{i\theta})^n = (\cos\theta + i\sin\theta)^n = e^{in\theta}$$

$$e^{i\theta}\cdot e^{i\theta}\cdot e^{i\theta}\cdot e^{i\theta}\cdot\ldots\cdot e^{i\theta} = e^{(i\theta+i\theta+i\theta+\ \ +i\theta)} = \underline{e^{in\theta}}$$

$$e^{in\theta} = \cos n\theta + i\sin n\theta = (\cos\theta \underline{+ i\sin}\theta)^n \qquad \underline{\text{DMF}}$$

$\underline{n=3}$ $\quad (\cos\theta + i\sin\theta)^3 = (\cos\theta + i\sin\theta)(\cos^2\theta + 2i\sin\theta\cos\theta + \underbrace{i^2}_{=-1}\sin^2\theta)$

$$= (\cos^3\theta + 2i\sin\theta\cos^2\theta - \cos\theta\sin^2\theta$$
$$i\sin\theta\cos^2\theta - 2\sin^2\theta\cos\theta - i\sin^3\theta)$$

$$= (\cos^3\theta - 3\cos\theta\sin^2\theta + i(3\sin\theta\cos^2\theta - \sin^3\theta))$$

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$$
$$\sin 3\theta = 3\sin\theta\cos^2\theta - \sin^3\theta$$

$$\underline{1} = \underline{\sqrt{1\cdot 1} = \sqrt{-1\cdot -1} = \sqrt{-1}\cdot\sqrt{-1} = i\cdot i} = \underline{-1}$$

Problem: Hva mener vi med $\sqrt{z}$?

Hvis z er et reelt positivt tall,

da er $\sqrt{z}$ det positive reelle tallet

som multiplisert med seg selv

blir z.

$$x^2 = 4$$
$$x = \underbrace{\sqrt{4}}_{=2} \text{ eller } x = \underbrace{-\sqrt{4}}_{=-2}$$

$\sqrt{a\cdot b} = \sqrt{a}\cdot\sqrt{b}$ dersom a og b er positive reelle tall.

$$\sqrt{-1\cdot -1} \overset{?}{=} \sqrt{-1}\cdot\sqrt{-1}$$

$$\underline{i}^2 = (\underline{-i})^2 = -1 \qquad z^2 = -1 \Leftarrow z = i \text{ eller } z = -i$$

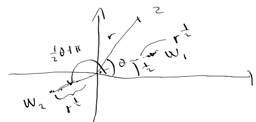
Def Dersom z er et komplekst tall så sier at w er en kvadratrot til z dersom $w^2 = z$.

$$z = re^{i\theta} = re^{i(\theta+2k\pi)} \qquad z^n = (re^{i\theta})^n = r^n e^{in\theta}$$

$n=\frac{1}{2}$ $$z^{\frac{1}{2}} = r^{\frac{1}{2}} e^{i\frac{1}{2}\theta}$$

$$= r^{\frac{1}{2}} e^{i\frac{1}{2}(\theta+2k\pi)} = r^{\frac{1}{2}} e^{i(\frac{1}{2}\theta + k\pi)}$$

$k = 0, 1, 2, \dots, -1, -2, \dots$



$k=1$ $$r^{\frac{1}{2}} e^{i(\frac{1}{2}\theta+\pi)}$$

$$w_1 = r^{\frac{1}{2}} e^{i\frac{1}{2}\theta}$$

$$w_2 = r^{\frac{1}{2}} e^{i(\frac{1}{2}\theta+\pi)}$$

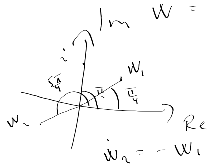
Både w_1 og w_2 er kvadratrøtter til z!

Eks

$z = i$ $\quad |z| = 1$ $\qquad z = a + ib$, $|z| = \sqrt{a^2+b^2}$

$i = e^{i\frac{\pi}{2}}$ $\qquad \operatorname{Arg}(i)$

$$w^2 = i$$
$$w^2 = e^{i(\frac{\pi}{2}+2k\pi)}$$
$$w = e^{i\frac{1}{2}(\frac{\pi}{2}+2k\pi)} = e^{i(\frac{\pi}{4}+k\pi)} \qquad k = 0, \pm 1, \pm 2, \dots$$



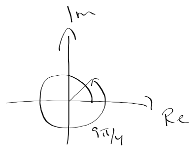
$k=0$: $e^{i\frac{\pi}{4}}$

$k=1$: $e^{i\frac{5\pi}{4}} = e^{i\frac{\pi}{4}+i\pi} = e^{i\frac{\pi}{4}} \cdot e^{i\pi}$ $\quad -1$

$k=2$: $e^{i\frac{\pi}{4}+2\pi} = e^{i\frac{\pi}{4}}$

$-e^{i\frac{\pi}{4}}$ $\qquad re^{i\theta}$

$w_2 = -w_1$



$$e^{i(\frac{\pi}{4}+2\pi)} = e^{i\frac{9\pi}{4}} = e^{i\frac{\pi}{4}}$$

$$z^2 + pz + q = 0 \qquad p, q \text{ komplekse tall}$$
$$z^2 + pz + (\tfrac{p}{2})^2 + q = \frac{p^2}{4}$$
$$(z + \tfrac{p}{2})^2 = \frac{p^2}{4} - q$$

$$z + \frac{p}{2} = r^{\frac{1}{2}} e^{i\frac{1}{2}(\theta+2k\pi)} \qquad \frac{p^2}{4} - q = re^{i\theta}$$

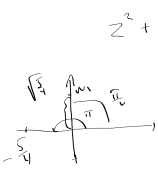
$$= \begin{cases} r^{\frac{1}{2}} e^{i\frac{1}{2}\theta} \\ r^{\frac{1}{2}} e^{i(\frac{1}{2}\theta+\pi)} \end{cases}$$

$$= \begin{cases} r^{\frac{1}{2}} e^{i\frac{1}{2}\theta} \\ -r^{\frac{1}{2}} e^{i\frac{1}{2}\theta} \end{cases}$$

$$z + \frac{p}{2} = \pm r^{\frac{1}{2}} e^{i\frac{1}{2}\theta}$$ der $\frac{p^2}{4} - q = re^{i\theta}$

$$z = -\frac{p}{2} \pm w_1$$ der $w_1^2 = \frac{p^2}{4} - q$

$$z^2 + iz + 1 = 0$$
$$z^2 + iz + (\tfrac{i}{2})^2 = -1 + (\tfrac{i}{2})^2 = -1 - \tfrac{1}{4} = -\tfrac{5}{4}$$



$$(z + \tfrac{i}{2})^2 = -\tfrac{5}{4} = (-1)\cdot(\tfrac{5}{4})$$

$$z + \frac{i}{2} = \begin{cases} i\sqrt{\frac{5}{4}} \\ -i\sqrt{\frac{5}{4}} \end{cases}$$

$$z = -\frac{i}{2} \pm i\,\frac{1}{2}\sqrt{5}$$
$$= \frac{-i \pm i\sqrt{5}}{2} = i\left(\frac{-1 \pm \sqrt{5}}{2}\right)$$