Komplekse tall og n-te røtter

$$z = a + ib = re^{i\theta} \qquad a, b \in \mathbb{R}$$
$$= r(\cos\theta + i\sin\theta) \quad r \geq 0 \quad 0 \leq \theta < 2\pi$$

$r = |z| = \sqrt{a^2 + b^2}$ $\quad a = \operatorname{Re} z \quad b = \operatorname{Im} z$

θ argumentet til z og $\quad \cos\theta = \frac{a}{r} \quad \sin\theta = \frac{b}{r}$

Def $\quad \underline{e^z = \overline{e^{(a+ib)}}} = e^a \cdot (\cos b + i\sin b)$

$$e^{i\theta} = \cos\theta + i\sin\theta \quad \leftarrow$$
$$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta)$$
$$= \cos\theta - i\sin\theta \quad \leftarrow$$
$$e^{i\theta} + e^{-i\theta} = 2\cos\theta \quad \Rightarrow \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$
$$e^{i\theta} - e^{-i\theta} = 2i\sin\theta \quad \Rightarrow \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Def:
$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$
$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

ex.
$$\cos i = \frac{e^{i \cdot i} + e^{-i \cdot i}}{2}$$
$$= \frac{e^{-1} + e^{1}}{2} = \frac{\frac{1}{e} + e}{2}$$
$$|\cos i| = \frac{\frac{1}{e} + e}{2} > 1$$

n-te røtter.

Def w er en n-te rot av z

dersom $\underline{w^n = z}$.

Spørsmål: Hvor mange n-te røtter har z?

$$z = r e^{i\theta}$$

Kan finne en n-rot ved

$$z^{\frac{1}{n}} = r^{\frac{1}{n}} e^{i \frac{1}{n}\theta}$$

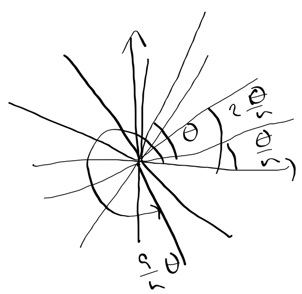
kalle denne n-rota for w_0

$$w_0 = r^{\frac{1}{n}} e^{i\frac{1}{n}\theta} \qquad \left(w_0^n = \left(r^{\frac{1}{n}} e^{i\frac{1}{n}\theta}\right)^n = r e^{i\theta} = z\right)$$

$$z = r e^{i(\theta + 2k\pi)} \qquad k = 0, \pm 1, \pm 2 \ldots$$

$$w_k = r^{\frac{1}{n}} e^{i\frac{1}{n}(\theta + 2k\pi)}$$

$$= r^{\frac{1}{n}} e^{i\left(\frac{\theta}{n} + \underbrace{2\frac{k}{n}\pi}\right)} \qquad 2\frac{k}{n}\pi = 2\pi \text{ når } k = n$$



$$\underbrace{w_0, w_1, w_2, \ldots, w_k, \ldots, w_{n-1}}$$

dette er de forskjellige n-te røttene til z

n-te røtter til $z = re^{i\theta}$:

$$w_0 = r^{\frac{1}{n}} e^{i\frac{\theta}{n}}, \quad w_1 = r^{\frac{1}{n}} e^{i\left(\frac{\theta}{n} + \frac{2}{n}\cdot\pi\right)}$$

$$\ldots \quad w_{n-1} = r^{\frac{1}{n}} e^{i\left(\frac{\theta}{n} + 2\frac{n-1}{n}\pi\right)}$$

$$z = re^{i\theta} \qquad w_k = r^{\frac{1}{n}} e^{i\left(\frac{\theta}{n} + 2\frac{k}{n}\pi\right)}$$

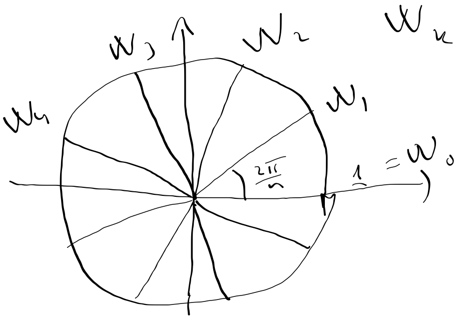
$$= \underbrace{r^{\frac{1}{n}} e^{i\frac{\theta}{n}}}_{w_0} \cdot \underbrace{e^{i\frac{2k}{n}\pi}}$$

$$z = 1 = e^{i2k\pi} \qquad k = 0, \pm 1, \ldots$$

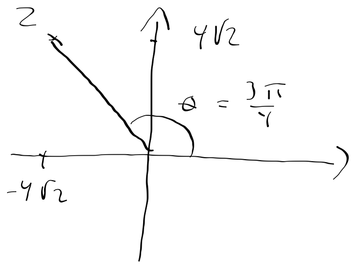
n-te røtter til 1:

$$w_0 = e^{i\,2\cdot\frac{0}{n}\cdot\pi} = 1$$

$$w_k = e^{i\,2\,\frac{k}{n}\pi} \qquad k = 1, 2, , n-1.$$



3-dje røttene til $z = -4\sqrt{2} + 4\sqrt{2}\, i$



$$r = |z| = \sqrt{(-4\sqrt{2})^2 + (4\sqrt{2})^2}$$
$$= \sqrt{32 + 32} = 8$$

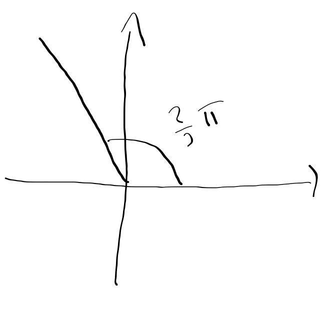
$$\left.\begin{aligned} \cos\theta &= \frac{-4\sqrt{2}}{8} = -\frac{\sqrt{2}}{2} \\ \sin\theta &= \frac{4\sqrt{2}}{8} = \frac{\sqrt{2}}{2} \end{aligned}\right\} \theta = \frac{3\pi}{4}$$

$$z = -4\sqrt{2} + 4\sqrt{2}\, i = \underline{8 e^{i \frac{3\pi}{4}}}$$

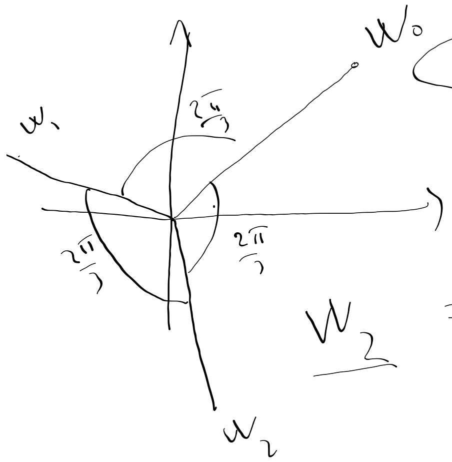
$$\underline{W_0} = 8^{\frac{1}{3}} \cdot e^{i \cdot \frac{1}{3} \cdot \frac{3\pi}{4}} = 2 e^{i\frac{\pi}{4}} = 2\left(\cos\frac{\pi}{4} + i \sin\frac{\pi}{4}\right)$$
$$= 2\left(\tfrac{1}{2}\sqrt{2} + i\,\tfrac{1}{2}\sqrt{2}\right)$$
$$= \underline{\underline{\sqrt{2} + i\sqrt{2}}}$$

$$w_1 = \left(8 e^{i\left(\frac{3\pi}{4} + 2\pi\right)}\right)^{\frac{1}{3}}$$
$$= 8^{\frac{1}{3}} e^{i\left(\frac{\pi}{4} + \frac{2}{3}\pi\right)} = 2 e^{i\left(\frac{11}{12}\pi\right)}$$

$$\underline{2 e^{i\frac{\pi}{4}}} \cdot e^{i\frac{2}{3}\pi} = \left(\sqrt{2} + i\sqrt{2}\right)\left(-\tfrac{1}{2} + i\,\tfrac{1}{2}\sqrt{3}\right)$$



$$\cos\tfrac{2}{3}\pi + i \sin\tfrac{2}{3}\pi$$
$$\left(-\tfrac{1}{2} + i\,\tfrac{1}{2}\sqrt{3}\right)$$



$$= \tfrac{1}{2}\left(\sqrt{2} + i\sqrt{2}\right)\left(-1 + i\sqrt{3}\right)$$
$$= \tfrac{1}{2}\left(-\sqrt{2} + i\sqrt{6} - i\sqrt{2} - \sqrt{6}\right)$$
$$= \tfrac{1}{2}\left(-\sqrt{2} - \sqrt{6} + i\left(\sqrt{6} - \sqrt{2}\right)\right)$$

$$\underline{W_2} = 2 e^{i\frac{\pi}{4}} e^{i\frac{4}{3}\pi} = \underline{2 e^{i\frac{19}{12}\cdot\pi}}$$