fra 9.5 grensesammenlikning og uegentlige integraler.

Setning: La $f, g : [a, \infty) \to \mathbb{R}$ være positive, kontinuerlige funksjoner.

Dersom $\int_a^\infty f(x)\,dx$ konvergerer og $\lim_{x\to\infty} \frac{g(x)}{f(x)} = L < \infty$, så konvergerer også $\int_a^\infty g(x)\,dx$.

Dersom $\int_a^\infty f(x)\,dx$ divergerer og $\lim_{x\to\infty} \frac{g(x)}{f(x)} > 0$, så divergerer også $\int_a^\infty g(x)\,dx$.

Bevis: Når $\lim_{x\to\infty} \frac{g(x)}{f(x)} = L$, $f, g > 0$ og f, g er kontinuerlige, så er $\frac{g(x)}{f(x)}$ begrenset på $[a, \infty)$

Altså: Det fins en C slik at $\frac{g(x)}{f(x)} < C$ for alle $x \in [a, \infty)$.

Dette er fordi $\frac{g}{f}$ er kontinuerlig og derfor begrenset på $[a, b]$ for hver $b > a$..

$\lim_{x\to\infty} \frac{g(x)}{f(x)} = L$ betyr at det fins en $b > a$ s.a. $\frac{g(x)}{f(x)} < L + 1$ når $x > b$.

på $[a, b]$ er $\frac{g}{f} < C'$. så

$$\frac{g(x)}{f(x)} < \underbrace{C' + L + 1}_{C} \quad \text{for hver } x \in [a, \infty)$$

$$\frac{g(x)}{f(x)} < C \Rightarrow g(x) < C \cdot f(x) \quad x \in [a, \infty)$$

så
$$\int_a^d g(x)\,dx < \int_a^d C f(x)\,dx \le \int_a^\infty C f(x)\,dx = C \cdot \int_a^\infty f(x)\,dx < \infty$$

$\Rightarrow \int_a^\infty g(x)\,dx$ konvergerer.

$\int_a^\infty f(x)\,dx$ konvergerer

eksempel:
$$\int_1^\infty \frac{3x^2+7}{4x^4+2x}\,dx$$

$$\int_1^\infty \frac{1}{x^2}\,dx \quad \text{konverger!}$$

ser på
$$\lim_{x\to\infty} \frac{\frac{3x^2+7}{4x^4+2x}}{\frac{1}{x^2}} = \lim_{x\to\infty} \frac{x^2(3x^2+7)}{4x^4+2x}$$

$$= \lim_{x\to\infty} \frac{(3x^4+7x^2)\frac{1}{x^4}}{(4x^4+2x)\frac{1}{x^4}} = \lim_{x\to\infty} \frac{3+\frac{7}{x^2}}{4+\frac{2}{x^3}} = \underline{\frac{3}{4}}$$

Av grensesammenlikning kan vi konkludere at
$$\int_1^\infty \frac{3x^2+7}{4x^4+2x}\,dx \quad \text{konvergerer!}$$

FLVA 1.1 - 2

n-tupler og vektorer

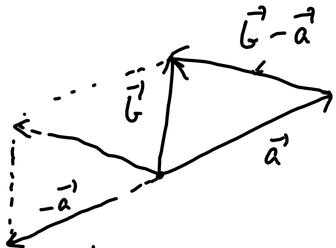
$\vec{a} = (a_1, a_2, \dots, a_n) \in \mathbb{R}^n \qquad a_i \in \mathbb{R}$

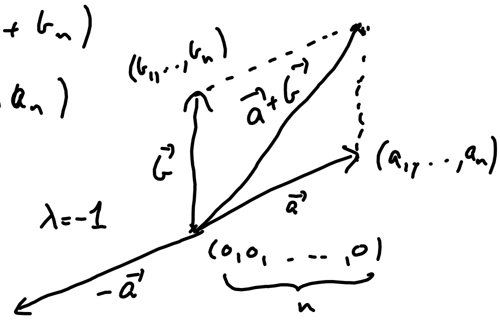
Regning med n-tupler (vektorer)

$\vec{a} = (a_1, a_2, \dots, a_n) \qquad \vec{b} = (b_1, b_2, \dots, b_n)$

$\vec{a} + \vec{b} = (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n)$

$\lambda \in \mathbb{R} \qquad \lambda \cdot \vec{a} = (\lambda a_1, \lambda a_2, \dots, \lambda a_n)$





indre produkt:

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + \dots + a_n b_n \in \mathbb{R}$$

$$\vec{a} \cdot \vec{a} = a_1^2 + a_2^2 + \dots + a_n^2 \geq 0$$

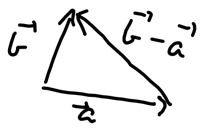
Def: $|\vec{a}| = \sqrt{\vec{a} \cdot \vec{a}} = \sqrt{a_1^2 + a_2^2 + \dots + a_n^2}$

$= 0$ bare hvis $\vec{a} = \vec{0}$.

Dersom $\vec{a} \cdot \vec{b} = a_1 b_1 + \dots + a_n b_n = \underline{0}$

sier vi at $\vec{a}$ og $\vec{b}$ er ortogonale.

Phytagoras' setning:



Dersom $\vec{a} \cdot \vec{b} = 0$

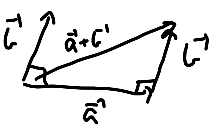
så er $|\vec{a}|^2 + |\vec{b}|^2 = |\vec{b} - \vec{a}|^2$.

Bevis: $|\vec{b} + \vec{a}|^2 = (b_1 + a_1)^2 + (b_2 + a_2)^2 + \dots + (b_n + a_n)^2$

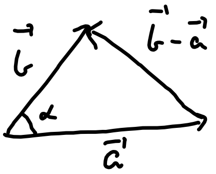
$= b_1^2 + 2a_1 b_1 + a_1^2 + \dots + b_n^2 + 2a_n b_n + a_n^2$

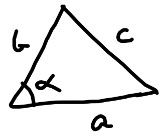
$= b_1^2 + b_2^2 + \dots + b_n^2 + a_1^2 + a_2^2 + \dots + a_n^2$

$+ 2\underbrace{(a_1 b_1 + \dots + a_n b_n)}_{=0}$



$= |\vec{a}|^2 + |\vec{b}|^2$





Cosinussetningen:

$$|\vec b-\vec a|^2 = |\vec a|^2+|\vec b|^2-2|\vec a||\vec b|\cdot\cos\alpha \qquad c^2 = a^2+b^2-2ab\cos\alpha$$

$$\| $$

$$b_1^2+b_2^2+\dots+b_n^2 + a_1^2+a_2^2+\dots+a_n^2 - 2(a_1b_1+\dots+a_nb_n) = a_1^2+\dots+a_n^2+b_1^2+\dots+b_n^2-2|\vec a||\vec b|\cos\alpha$$

$$\vec a\cdot\vec b = |\vec a||\vec b|\cdot\cos\alpha$$

$$\cos\alpha = \frac{\vec a\cdot\vec b}{|\vec a||\vec b|}$$

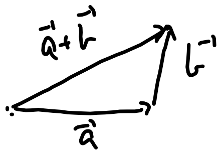
$\vec a$ og $\vec b$ er ortogonale hvis og bare hvis $\alpha = \frac{\pi}{2}$.

$\vec a = (1, 0, 2, 2)$ hva e vinkelen α mellom vektorene?

$\vec b = (-2, 2, 1, 4)$

$$\cos\alpha = \frac{\vec a\cdot\vec b}{|\vec a||\vec b|} = \frac{-2+0+2+8}{\sqrt{1+4+4}\sqrt{4+4+1+16}} = \frac{8}{3\cdot 5} = \frac{8}{15}$$

$$\alpha = \text{Arccos}\,\frac{8}{15}$$



$$|\vec a+\vec b| \overset{?}{\leq} |\vec a|+|\vec b|$$

$$|\vec a+\vec b|^2 = a_1^2+a_2^2+\dots+a_n^2$$
$$+b_1^2+b_2^2+\dots+b_n^2$$
$$+2(a_1b_1+a_2b_2+\dots+a_nb_n)$$
$$= |\vec a|^2+|\vec b|^2+2\,\vec a\cdot\vec b$$
$$= |\vec a|^2+|\vec b|^2+2|\vec a||\vec b|\cos\alpha$$
$$\leq |\vec a|^2+|\vec b|^2+2|\vec a||\vec b|$$
$$= (|\vec a|+|\vec b|)^2.$$

$$\vec a\cdot\vec b = |\vec a|\cdot|\vec b|\cos\alpha$$

Hva med vektorprodukt (kryssprodukt)

$\vec{a} \in \mathbb{R}^3 \ni \vec{b}$

$\vec{a} = (a_1, a_2, a_3)$ $\quad$ $\vec{b} = (b_1, b_2, b_3)$

$$\vec{a} \times \vec{b} = (a_2 b_3 - a_3 b_2, \; a_3 b_1 - a_1 b_3, \; a_1 b_2 - a_2 b_1)$$

Komplekse n-tupler.

$\vec{a} = (a_1, a_2, \dots, a_n)$ $\quad$ $a_i \in \mathbb{C}$ $\quad$ $a_j = r_j + i s_j$

$\vec{b} = (b_1, b_2, \dots, b_n)$ $\quad$ $b_i \in \mathbb{C}$ $\quad$ $b_j = u_j + i v_j$

$\vec{a} + \vec{b} = (a_1 + b_1, \dots, a_n + b_n)$ $\quad$ $|a| = \sqrt{a \cdot \bar{a}}$

$\lambda \vec{a} \quad (\lambda a_1, \quad, \lambda a_n)$

Def: $\vec{a} \cdot \vec{b} \stackrel{\Delta}{=} (a_1 \bar{b}_1 + a_2 \bar{b}_2 + \dots + a_n \bar{b}_n)$

$\vec{a} \cdot \vec{a} = a_1 \bar{a}_1 + \quad + a_n \bar{a}_n \in \mathbb{C}$ (ikke nødvendigvis reelt eller positivt).

$\geq 0 \quad \in \mathbb{R}$

$\vec{a} \cdot \vec{a} \geq 0$ et reelt tall

$= 0$ bare hvis $\vec{a} = \vec{0}$.

Def: $|\vec{a}| \stackrel{\Delta}{=} \sqrt{\vec{a} \cdot \vec{a}}$