Hvorfor komplekse tall?

De gir mulighet for å regne med kvadrøtter til negative tall.

Det vil si også regning med n-te røtter, og røtter til n-te gradslikninger.

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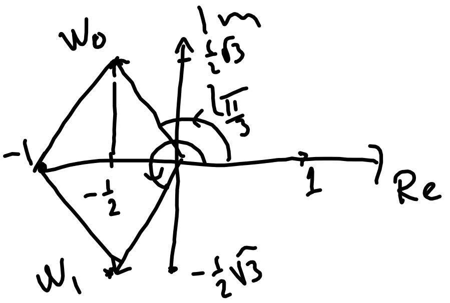
eks: $x^2 + x + 1 = 0$

ABC-formelen:
$$x = \frac{-1 \pm \sqrt{1^2 - 4}}{2}$$
$$= \frac{-1 \pm \sqrt{-3}}{2} \qquad i^2 = -\underline{1}$$



$$= \frac{-1 \pm i\sqrt{3}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$
$$w_0 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$
$$w_1 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$

$$\underline{x^2 + x + 1} = (\overset{w_0}{x} - w_0)(x - w_1) \leftarrow$$
$$= \underline{x^2 - (w_0 + w_1)x + w_0 w_1}$$

$$\underline{w_0 + w_1 = -\underline{1}}$$
$$\underline{w_0 w_1 = \underline{1}}$$

$$|w_0| = \sqrt{\left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\sqrt{3}\right)^2} = \sqrt{\tfrac{1}{4} + \tfrac{3}{4}} = \underline{1}$$

$$= \underline{\left(x + \frac{1}{2} - i\frac{\sqrt{3}}{2}\right)\left(x + \frac{1}{2} + i\frac{\sqrt{3}}{2}\right)}$$

Algebraens fundamentalsetning:

La $c_0, c_1, \dots, c_n^{\neq 0}$ være komplekse tall

og la $P(z) = \underline{c_n} z^n + c_{n-1} z^{n-1} + \dots + c_1 z + c_0$.

Da fins det komplekse tall

$r_1, r_2, \dots, r_n$ slik at

$$P(z) = \underline{c_n} (z - r_1)(z - r_2) \cdots (z - r_n).$$

$\underline{r_1, r_2, \dots, r_n}$ er røtter (løsninger)

til likningen $P(z) = 0$

d.v.s. $c_n z^n + c_{n-1} z^{n-1} + \dots + c_1 z + c_0 = 0$.

z_0 er rot dersom $P(z_0) = 0$. $\underline{P(r_k)} = c_n (r_k - r_1) \cdots \overset{0}{\overset{\shortparallel}{(r_k - r_k)}} \cdots (r_k - r_n)$

$\underline{= 0}$

Hvis $s \neq r_1, r_2, \dots r_n$,

da er $P(s) = \overset{0}{\underset{\#}{c_n}} \overset{0}{\underset{\#}{(s - r_1)}} \overset{0}{\underset{\#}{(s - r_2)}} \cdots \overset{0}{\underset{\#}{(s - r_n)}} \neq 0$

så s er ikke rot!

Så $r_1 \dots r_n$ er nøyaktig røttene til

$P(z) = 0$.

Korollar: En n-te grads likning med komplekse koeffisienter har n røtter (løsninger).

$$x^2 - 2ix - 1 = 0$$

ABC. $\underline{x} = \dfrac{2i \pm \sqrt{4i^2 + 4}}{2}$ $(2i)^2$

$$= \frac{2i \pm \sqrt{4(-1)+4}}{2} = \frac{2i}{2} = \underline{i}$$

$$\underline{x^2 - 2ix - 1 = (x-i)^2}$$

$$(x-i)^n$$

I faktoriseringen av $P(z)$ kan vi samle like faktorer:

$$P(z) = c_n z^n + \dots + c_1 z + c_0$$
$$= c_n(z-r_1)(z-r_2)\dots(z-r_n)$$
$$= c_n(z-r_1)^{n_1}(z-r_2)^{n_2}\dots(z-r_j)^{n_j}$$

$n_1 + n_2 + \dots + n_j = n$

Hvis $r_1, r_2 \dots r_j$ alle er forskjellige, kaller vi n_i for <u>multiplisiteten</u> til r_i i $P(z)$. $1 \le i \le j$

$$\boxed{z^n = a + ib}$$ har n løsn.

$$z^n = re^{i\theta}$$

$$w_0 = r^{\frac{1}{n}} e^{i\frac{\theta}{n}}, \quad w_1 = r^{\frac{1}{n}} e^{i\left(\frac{\theta}{n} + \frac{2\pi}{n}\right)}$$
$$\vdots$$
$$w_k = r^{\frac{1}{n}} e^{i\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)}$$

<u>n- røtter</u> $k = 0, 1, 2, \dots, n-1$

$$z^n - re^{i\theta}$$
$$= (z - w_0)(z - w_1)\dots(z - w_{n-1})$$
$$= \left(z - r^{\frac{1}{n}} e^{i\frac{\theta}{n}}\right)\left(z - r^{\frac{1}{n}} e^{i\left(\frac{\theta}{n} + \frac{2\pi}{n}\right)}\right)\dots\left(z - r^{\frac{1}{n}} e^{i\left(\frac{\theta}{n} + 2\frac{n-1}{n}\pi\right)}\right)$$

$$\parallel$$

$$z - r^{\frac{1}{n}} e^{i\left(\frac{\theta}{n} + 2\frac{n-1}{n}\pi\right)}$$

røttene i $z^3 - 11 = 0$

er 3-dje røtter til 11

røttene i $z^2 + 2z + 3$ er

ospå røtter.

Ser nå på likningen med reelle koeffisienter.

$$P(x) = c_n x^n + c_{n-1} x^{n-1} \dots c_1 x + c_0 = 0$$

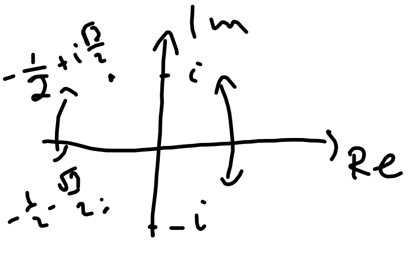
$$\underline{c_i \in \mathbb{R}.} \quad i = 0, \dots, n$$

A.F.: $P(x) = c_n (x - r_1) \dots (x - r_n)$ for komplekse tall $r_1 \dots r_n$.

$$\underline{x^2 + 1} = \underline{(x - \underline{i})(x + \underline{i})} \qquad \pm i$$

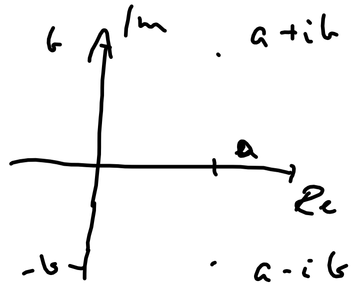
$$\underline{x^2 + x + 1} = \left(x + \tfrac{1}{2} - i\tfrac{\sqrt{3}}{2}\right)\left(x + \tfrac{1}{2} + i\tfrac{\sqrt{3}}{2}\right)$$

$$-\tfrac{1}{2} \pm i\tfrac{\sqrt{3}}{2}$$



Den komplekse konjugerte til $z = a + ib$

er $\bar{z} = a - ib$



$z = a+ib \qquad \bar{z} = a-ib$

$w = c+id \qquad \bar{w} = c-id$

$$\overline{z+w} = \overline{(a+ib+c+id)} = \overline{(a+c+i(b+d))}$$
$$= a+c-i(b+d)$$
$$= a+c-ib-id = a-ib+c-id$$
$$= \bar{z} + \bar{w}$$

$$\overline{zw} = \overline{(a+ib)(c+id)} = \overline{(ac-bd+i(bc+ad))}$$
$$= \underline{ac-bd-i(bc+ad)}$$

$$\bar{z}\cdot\bar{w} = (a-ib)(c-id) = \underline{ac-bd-i(bc+ad)}$$

så $\underline{\overline{zw} = \bar{z}\cdot\bar{w}}$.

Lemma: Hvis w er en rot i $P(z) = 0$ der $P(z)$ bare har reelle koeffisienter, så er også $\bar{w}$ en rot i $P(z) = 0$.

Bevis Vet $\underline{P(w)} = c_n w^n + c_{n-1} w^{n-1} + \quad + c_1 w + c_0 = \underline{0}$

Vil vise at da er $P(\bar{w}) = 0$!

$$\underline{P(\bar{w})} = c_n \bar{w}^n + c_{n-1}\bar{w}^{n-1} + \quad + c_1\bar{w} + c_0 =$$
$$= c_n \overline{w^n} + c_{n-1}\overline{w^{n-1}} + \quad + c_1\bar{w} + c_0 =$$

$c_i \in \mathbb{R}$, $\bar{c}_i = c_i$

$$= \bar{c}_n \overline{w^n} + \bar{c}_{n-1}\overline{w^{n-1}} + \quad + \bar{c}_1\bar{w} + \bar{c}_0 =$$
$$= \overline{c_n w^n} + \overline{c_{n-1}w^{n-1}} + \quad + \overline{c_1 w} + \bar{c}_0$$
$$= \overline{c_n w^n + \quad + c_1 w + c_0} = \overline{P(w)} = \bar{0} = \underline{0}$$

så $\bar{w}$ er også rot.

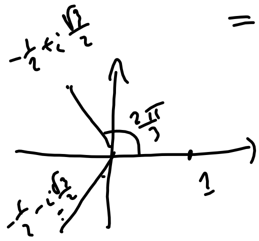
Korollar: De komplekse røttene til et reelt polynom opptrer i konjugerte par.

Så: $P(z) = c_n(z-r_1)(z-\bar{r}_1)\cdots(z-r_k)(z-\bar{r}_k)$
$(z-r_{k+1})\cdots(z-r_s)$

$r_{k+1}, \dots, r_s \in \mathbb{R}$

eks

$$z^3 - 1$$
$$= (z-1)\left(z+\tfrac{1}{2}+i\tfrac{\sqrt{3}}{2}\right)\left(z+\tfrac{1}{2}-i\tfrac{\sqrt{3}}{2}\right)$$



$$\underbrace{z^5 - 3z^4 + 2z^3 - z^2 + 1}_{(z-r_1)(z-\bar{r}_1)(z-r_2)(z-\bar{r}_2)(z-r_3)} = 0$$

Setning: Et reelt polynom av odde grad har minst en reell rot!

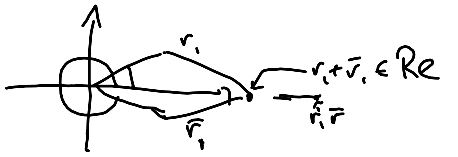
Setningen: Konjugerte røtter i et reelt polynom har samme multiplisitet.

$$P(z) = c_n z^n + \dots + c_1 z + c_0 \qquad c_i \in \mathbb{R}$$
$$= c_n (z-r_1)^{n_1}(z-\bar{r}_1)^{n_1'}(z-r_2)^{n_2}(z-\bar{r}_2)^{n_2'}$$
$$\cdots (z-r_j)^{n_j}(z-\bar{r}_j)^{n_j'}(z-r_{j+1})\cdots(z-r_s)$$

Vil vise at
$n_1 = n_1'$ $n_2 = n_2'$ $\dots$ $n_j = n_j'$ $\qquad r_{j+1}\dots r_s \in \mathbb{R}$

$$(z-r_1)(z-\bar{r}_1) = z^2 - (r_1+\bar{r}_1)z + r_1\bar{r}_1$$

er et reelt polynom!



$$\frac{P(z)}{z^2-(r_1+\bar{r}_1)z+r_1\bar{r}} \quad \text{er et reelt polynom}$$

$\Rightarrow$

$$P(z) = (z^2-(r_1+\bar{r}_1)z+r_1\bar{r}_1)^{n_1}\cdots$$
$$\cdots(z^2-(r_j+\bar{r}_j)z+r_j\bar{r}_j)^{n_j}(z-r_{j+1})\cdots$$
$$\cdots(z-r_s)$$