Vektorprodukt, volumberegning og matriser

vektorprodukt har vi bare for vektorer i $\mathbb{R}^3$.

$$\vec{a} = (a_1, a_2, a_3) \qquad \vec{b} = (b_1, b_2, b_3)$$

Def. $\vec{a} \times \vec{b} = (a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1)$

$$\vec{i} = (1, 0, 0) \qquad \vec{j} = (0, 1, 0) \qquad \vec{k} = (0, 0, 1)$$

$$\vec{a} = a_1 \cdot \vec{i} + a_2 \vec{j} + a_3 \vec{k}$$

$$\vec{b} = b_1 \vec{i} + b_2 \vec{j} + b_3 \vec{k}$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \vec{i}\,(a_2 \cdot b_3 - a_3 b_2) + \vec{j}\,(a_3 b_1 - a_1 b_3) + \vec{k}\,(a_1 b_2 - a_2 b_1)$$

$$\rightarrow \quad \vec{b} \times \vec{a} = -(\vec{a} \times \vec{b})$$

$$\shortparallel \quad \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} \quad ?$$

Regneregler:

- $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$.
- $\lambda(\vec{a} \times \vec{b}) = (\lambda \vec{a}) \times \vec{b} = \vec{a} \times (\lambda \vec{b})$.

$$\vec{a} \times (\vec{b} \times \vec{c}) \overset{?}{\neq} (\vec{a} \times \vec{b}) \times \vec{c}$$

$$\big((1,1,0) \times (1,0,0)\big) \times (0,0,1) = (0,0,0)$$

$$(1,1,0) \times \big((1,0,0) \times (0,0,1)\big) = (0,0,-1)$$

$$\begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = \vec{i}(1\cdot 0 - 0 \cdot 0) + \vec{j}(0 \cdot 1 - 1 \cdot 0) + \vec{k}(1 \cdot 0 - 1 \cdot 1)$$

$$= -\vec{k} = (0,0,-1)$$

$$(0,0,-1) \times (0,0,1) = \begin{vmatrix} i & j & k \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{vmatrix} = (0,0,0)$$

$$(0,0,0) \times (0,0,-1) = \begin{vmatrix} i & j & k \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{vmatrix} = (0,0,0)$$

$$\begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = (0,-1,0)$$

$$(1,1,0) \times (0,-1,0)$$

$$= \begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ 0 & -1 & 0 \end{vmatrix} = (0,0,-1) = \underline{-\vec{k}}$$

$\vec{a}=(a_1, a_2, a_3)$ $\vec{b}=(b_1, b_2, b_3)$ $\vec{c}=(c_1, c_2, c_3)$

$$\vec{a}\cdot(\vec{b}\times\vec{c}) \stackrel{?}{=} (a_1, a_2, a_3)\cdot\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$(\vec{a}\times\vec{b})\cdot\vec{c}$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1).$$

$||$

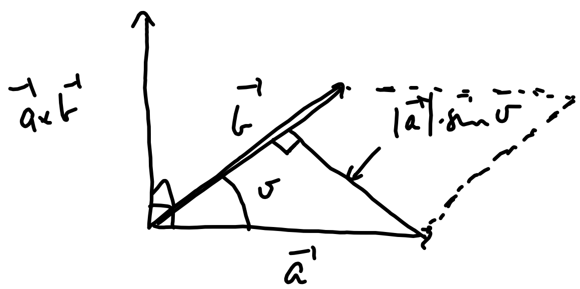
$$(\vec{a}\times\vec{b})\cdot\vec{c} = \vec{c}\cdot(\vec{a}\times\vec{b})$$

$$= \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = c_1(a_2b_3 - a_3b_2) + c_2(a_3b_1 - a_1b_3) + c_3(a_1b_2 - a_2b_1)$$

$$(\vec{a}\times\vec{b})\cdot\vec{a} = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \cancel{a_1(a_2b_3} - \cancel{a_3b_2)} + a_2(a_3b_1 - \cancel{a_1b_3}) + a_3(\cancel{a_1b_2} - a_2b_1) = 0$$

$\vec{a}\times\vec{b}$ er ortogonal til $\vec{a}$.

$\vec{a}\times\vec{b}$ er ortogonal til $\vec{b}$



$$|\vec{a}\times\vec{b}| = |\vec{a}|\cdot|\vec{b}|\cdot \sin v \quad (*)$$

skal vise dette!

eks $|\vec{a}\times\vec{b}|$ = areal til parallellogrammet utspent av $\vec{a}$ og $\vec{b}$

For å vise $(*)$ viser vi først

$$|\vec{a}\times\vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2 - (a\cdot b)^2 \quad \text{Lagrange}$$

$$|\vec{a}\times\vec{b}|^2 = (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2$$

$$= |\vec{a}|^2|b|^2 - (a\cdot b)^2 = (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2$$

$a_1^2 b_2^2$

Lagrange

$$|\vec{a}\times\vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2$$

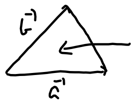
$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cdot\cos v$

$$= |\vec{a}|^2|\vec{b}|^2 - |\vec{a}|^2|\vec{b}|^2\cos^2 v$$

$$= |\vec{a}|^2|\vec{b}|^2(1-\cos^2 v)$$

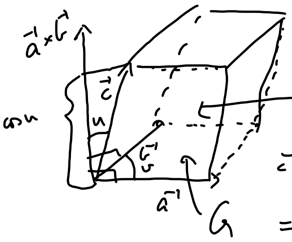
$$= |\vec{a}|^2|\vec{b}|^2\sin^2 v$$

arealet $= |\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin v$ □



arealet $= \frac{1}{2}|\vec{a}\times\vec{b}|$

Volum:



$h = |\vec{c}|\cdot\cos u$

Volumet $= G\cdot h$

$$= |\vec{a}\times\vec{b}|\cdot|\vec{c}|\cdot\cos u$$

$$= |(\vec{a}\times\vec{b})\cdot\vec{c}|$$

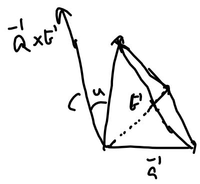
Setning: Volumet av parallellepipedet utspent av $\vec{a}, \vec{b}, \vec{c}$ er

$|(\vec{a}\times\vec{b})\cdot\vec{c}|$. □

$$|(\vec{a}\times\vec{b})\cdot\vec{c}| = |\vec{a}\cdot(\vec{b}\times\vec{c})| = |(\vec{a}\times\vec{c})\cdot\vec{b}|$$

$$= \left|\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}\right|$$

Den trekanta pyramiden utspent av $\vec{a}, \vec{b}, \vec{c}$ har volum $= \frac{1}{6}|(\vec{a}\times\vec{b})\cdot\vec{c}|$.

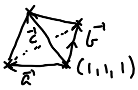


$$\text{Volum} = \frac{1}{3}G\cdot h$$

$$= \frac{1}{3}\cdot\frac{1}{2}|\vec{a}\times\vec{b}|\underbrace{|\vec{c}|\cdot u}_{h}$$

$$= \left|\frac{1}{6}(\vec{a}\times\vec{b})\cdot\vec{c}\right|.$$

Finn volumet til den trekantete pyramiden med hjørner i $(2,0,0)$, $(1,1,1)$, $(0,1,2)$ og $(1,2,0)$.



$\vec{a} = (2,0,0) - (1,1,1) = (1,-1,-1)$

$\vec{b} = (0,1,2) - (1,1,1) = (-1,0,1)$

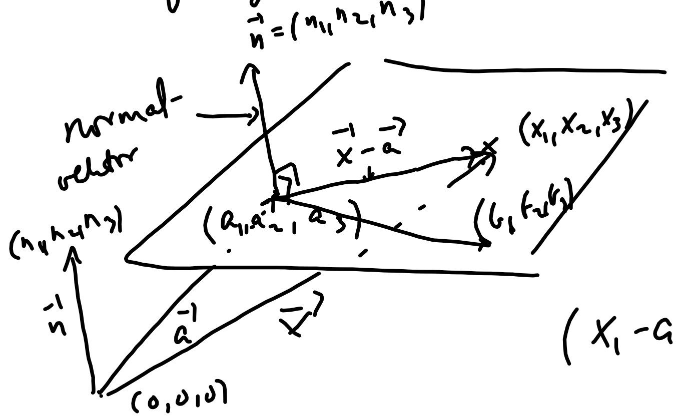
$\vec{c} = (1,2,0) - (1,1,1) = (0,1,-1)$

Volumet er $\frac{1}{6}|(\vec{a}\times\vec{b})\cdot\vec{c}|$

$$= \frac{1}{6}\left|\begin{vmatrix} 1 & -1 & -1 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{vmatrix}\right| = \frac{1}{6}\Big|\big(1\cdot(0\cdot(-1)-1\cdot 1) + (-1)(1\cdot 0-(-1)(-1)) + (-1)(-1\cdot 1 - 0\cdot 0)\big)\Big|$$

$$= \frac{1}{6}\big|(-1+1+1)\big| = \frac{1}{6}$$

Ligning til et plan i $\mathbb{R}^3$ (koordinater x_1, x_2, x_3).



$$\underline{(\vec{x} - \vec{a}) \cdot \vec{n} = 0}$$

$$\parallel$$

$$(x_1 - a_1,\ x_2 - a_2,\ x_3 - a_3) \cdot (n_1, n_2, n_3)$$

$$= \underline{n_1(x_1 - a_1) + n_2(x_2 - a_2) + n_3(x_3 - a_3) = 0}$$

$$\boxed{n_1 x_1 + n_2 x_2 + n_3 x_3 = n_1 a_1 + n_2 a_2 + n_3 a_3}$$

$$\underline{= n_1 b_1 + n_2 b_2 + n_3 b_3}$$

$$\vec{n} \cdot \vec{a} - \vec{n} \cdot \vec{b}$$
$$= \vec{n} \cdot (\vec{a} - \vec{b}) = 0$$