1.5-8 Matriser, matriseregning og determinanter

En matrise er et rektangulært oppsett av tall.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \cdots & & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}^{m\times n} \begin{matrix} \leftarrow \text{rad} \\ \\ \\ \leftarrow \end{matrix}$$

a_{ij} reelle (eller komplekse) tall.

↑ søylene

A er en $(m\times n)$-matrise,

A har m <u>rader</u> og n <u>søyler</u>

a_{ij} kalles <u>elementet</u> som ligger i den i-te raden og den j-te søylen

En $(1\times n)$-matrise kalles en rad-vektor

En $(m\times 1)$-matrise kalles en søyle-vektor.

Regne med matriser:

$$A^{(m\times n)} + B^{(r\times s)} = C^{(m\times n)}$$

mulig bare når $m=r$ og $n=S$

eks $\cancel{\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}} =$

$$(a,b,c) + (d,e,f) = (a+d, b+e, c+f)$$

$$\begin{pmatrix} a_{ij} \end{pmatrix}^{m\times n} + \begin{pmatrix} b_{ij} \end{pmatrix}^{m\times n} = \begin{pmatrix} c_{ij} \end{pmatrix}^{m\times n}$$

eks $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} + \begin{pmatrix} 0 & 2 & -1 \\ -1 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 4 & 2 \\ 3 & 5 & 9 \end{pmatrix}$ $\quad c_{ij} = a_{ij} + b_{ij}$

$$cA = c\begin{pmatrix} a_{ij} \end{pmatrix} = \begin{pmatrix} ca_{ij} \end{pmatrix}$$

c et tall

Transponering:

$A^T = (a_{ji})$ (A transponert)

dersom $A = (a_{ij})$

$$A^{(m\times n)} = (a_{ij}) \quad \begin{matrix} 1 \le i \le m \\ 1 \le j \le n \end{matrix} \qquad (A^T)^{(n\times m)} = (a_{ji}) \quad \begin{matrix} 1 \le j \le n \\ 1 \le i \le m \end{matrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

Multiplikasjon

$$A^{m\times n} \cdot B^{n\times p} = C^{m\times p}$$

$$\begin{pmatrix} \underline{a_{11}\ a_{12} \cdots a_{1n}} \\ \vdots \\ a_{m1} \quad a_{mn} \end{pmatrix}^{m\times n} \begin{pmatrix} b_{11} & \cdots & b_{1p} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{np} \end{pmatrix}^{n\times p} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{1n}b_{n1} & \cdots & c_{1p} \\ \vdots & & c_{ij} \\ \vdots & & \end{pmatrix}$$

(c_{ij})

$$(a_1 \quad , a_n)^{1\times n} \cdot \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}^{n\times 1} = (a_1 b_1 + \quad + a_n b_n)^{1\times 1}$$

$$\underline{c_{ij}} = (a_{i1}\ a_{i2} \cdots a_{in}) \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{pmatrix} = \underline{a_{i1} b_{1j} + \quad + a_{in} b_{nj}}$$

$$\underline{c_{1p} = a_{11} b_{1p} + \quad + a_{1n} b_{np}}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}^{2\times 3} \cdot \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}^{3\times 2} = \begin{pmatrix} 14 & 32 \\ 32 & 77 \end{pmatrix}^{2\times 2}$$

$$A(B_1 + B_2) = AB_1 + AB_2$$

$$A^{m\times n}(B^{n\times p}C^{p\times r}) = ((AB)C)^{m\times r}$$

$$A^{m\times n} B^{n\times p} \overset{?}{\neq} B^{n\times p} A^{m\times n}$$

gir bare mening når $p=n$ og $m=n$.

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\neq$$

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

$$A^{m\times n} = (\underline{a_{ij}})$$

$$\| A \| = \sqrt{\sum_j \sum_i a_{ij}^2}$$

$$\underline{\underline{A}}^{m\times n} \ \underline{X}^{n\times 1} = \underline{Y}^{m\times 1} \qquad X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad Y = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}$$

$$y_i = (a_{i1} \cdots a_{in}) \cdot \begin{pmatrix} x_1 \\ \\ x_n \end{pmatrix}$$

$$|y_i| \leq |a_{i1}| \cdot |x_1| + \quad + |a_{in}||x_n|$$

$$|y_i|^2 \leq \underline{(|a_{i1}|^2 + \quad + |a_{in}|^2)(|x_1|^2 +} \quad + |x_n|^2)$$

$$|Y|^2 = \sum_i |y_i|^2 \leq \| A \|^2 \, |x|^2$$

$$\underline{|Y| \leq \| A \| \cdot |x|} \ \leftarrow$$

Kvadratiske matriser.

$$A^{n\times n} \Rightarrow (A^T)^{n\times n}$$

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \qquad \begin{pmatrix} a_{11} & & a_{n1} \\ & \ddots & \\ a_{1n} & & a_{nn} \end{pmatrix}$$

$A = A^T \rightarrow A$ er symmetrisk.

$$I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Gitt $A^{n\times n}$ fins det en $B^{n\times n}$ s.a

$$A^{n\times n} \cdot B^{n\times n} = I \quad ?$$

$$\underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} x & y \\ z & w \end{pmatrix}}_{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}} \overset{?}{=} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x & y \\ z & w \end{pmatrix} \overset{?}{=} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} x+z & y+w \\ x & y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\left.\begin{array}{l} x+z=1 \\ y+w=0 \\ x=0 \\ y=1 \end{array}\right\} \quad \begin{array}{l} x=0 \\ y=1 \\ z=1 \\ w=-1 \end{array} \quad \text{så} \quad \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

<u>Setning</u> Dersom det fins en $X^{n\times n}$ slik at $A^{n\times n} \cdot X^{n\times n} = I^{n\times n}$ så er også $X \cdot A = I$.

Denne X kalles <u>invers</u> matrisen til A.

Dersom A ikke har en invers, sier vi at A er singulær.

<u>eksempel</u>

$$\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} x+z & y+w \\ 2x+2z & 2y+2w \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\left.\begin{array}{l} \rightarrow x+z=1 \Rightarrow 2x+2z=2 \\ y+w=0 \\ \rightarrow 2x+2z=0 \\ 2y+2w=1 \end{array}\right\} \text{umulig}$$

$\Rightarrow \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$ er singulær.

<u>Determinanter</u> til kvadratiske matriser.

se på (2×2) og (3×3) matriser.

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \underline{a_{11}a_{22} - a_{12}a_{21}}$$

$$|A| = \left|\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\right| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = \underline{\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}}$$

$\vec{a} = (a_1 \quad a_2 \quad a_3)$
$\vec{b} = (b_1 \quad b_2 \quad b_3)$
$\vec{c} = (c_1 \quad c_2 \quad c_3)$

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$$

er singulær. (hvis og bare hvis determinanten er 0)

$$|A| = \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 1 \cdot 2 - 1 \cdot 2 = \underline{\underline{0}}$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad |A| = ad - bc$$

$$\begin{vmatrix} c & d \\ a & b \end{vmatrix} = cb - ad = - \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$\begin{aligned} \begin{vmatrix} a & b \\ c+a & d+b \end{vmatrix} &= a(d+b) - b(c+a) \\ &= ad - bc + ab - ab \\ &= ad - bc = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \end{aligned}$$