THE AXIOM OF DEPENDENT CHOICE

Simon Foldvik

15. February 2020

Theorem 1. *Suppose X is a non-empty set and R a relation on X such that there for every $u \in X$ exists $v \in X$ such that uRv. If $a \in X$, there exists a sequence $(u_k)_{k \in \mathbb{N}}$ in X such that $u_0 = a$ and*

$$u_k R u_{k+1} \quad \textit{for all} \quad k \in \mathbb{N}.$$

Proof. For $u \in X$ define

$$R_u \coloneqq \{v \in X : uRv\}.$$

The family $(R_u)_{u \in X}$ consists of non-empty sets; let

$$f \colon X \to \bigcup_{u \in X} R_u$$

be a choice function, that is, suppose $f(u) \in R_u$ for every $u \in X$. It may happen that $\bigcup_{u \in X} R_u \neq X$, hence we let $g \colon X \to X$ coextend f by defining $g(u) = f(u)$ for all $u \in X$. If $a \in X$, we can apply the recursion theorem with g to obtain a sequence $(u_k)_{k \in \mathbb{N}}$ in X such that $u_0 = a$ and $u_{k+1} = g(u_k)$ for all $k \in \mathbb{N}$. Since for all $u \in X$ we have $g(u) \in R_u$, it follows that

$$u_k R u_{k+1} \quad \text{for all} \quad k \in \mathbb{N}. \qquad \blacksquare$$