BASIC PROPERTIES OF LIMIT INFERIOR AND LIMIT SUPERIOR

Before we consider the limit inferior and the limit superior, we establish the following characterization of sequence convergence.

Proposition 1. *Suppose that u is an extended real number and that $(u_n)_{n=0}^{\infty}$ is a sequence of extended real numbers. Then $(u_n)_{n=0}^{\infty}$ converges to u if and only if every subsequence of $(u_n)_{n=0}^{\infty}$ converges to u.*

Definition 2. Let $(u_n)_{n=0}^{\infty}$ be a sequence of extended real numbers. Then the sequences $(\inf_{m\geq n} u_m)_{n=0}^{\infty}$ and $(\sup_{m\geq n} u_m)_{n=0}^{\infty}$ are increasing and decreasing, respectively, hence convergent in the extended real number system. We define

$$\liminf_{n\to\infty} u_n = \lim_{n\to\infty} \big(\inf_{m\geq n} u_m\big) = \sup_{n\in\mathbf{N}} \big(\inf_{m\geq n} u_m\big), \tag{1}$$

called the *limit inferior* of the sequence $(u_n)_{n=0}^{\infty}$, and we define

$$\limsup_{n\to\infty} u_n = \lim_{n\to\infty} \big(\sup_{m\geq n} u_m\big) = \inf_{n\in\mathbf{N}} \big(\sup_{m\geq n} u_m\big), \tag{2}$$

called the *limit superior* of the sequence $(u_n)_{n=0}^{\infty}$.

In (1) and (2), the first equality is a matter of definition; the second equality follows from the monotone convergence of the sequences of infima and suprema.

Proposition 3. *Suppose that $(u_n)_{n=0}^{\infty}$ is a sequence of extended real numbers. Then one has*

$$\inf_{n\in\mathbf{N}} u_n \leq \liminf_{n\to\infty} u_n \leq \limsup_{n\to\infty} u_n \leq \sup_{n\in\mathbf{N}} u_n.$$

In particular, if there exists a real number $M \geq 0$ such that $|u_n| \leq M$ holds for every n, then both the limit inferior and the limit superior of $(u_n)_{n=0}^{\infty}$ is a real number.

Proposition 4. *If $(u_n)_{n=0}^{\infty}$ and $(v_n)_{n=0}^{\infty}$ are two sequences of extended real numbers such that $u_n \leq v_n$ holds for every n, one has*

$$\liminf_{n\to\infty} u_n \leq \liminf_{n\to\infty} v_n \quad \text{and} \quad \limsup_{n\to\infty} u_n \leq \limsup_{n\to\infty} v_n.$$

Proposition 5. *Consider a sequence $(u_n)_{n=0}^{\infty}$ of extended real numbers, and let α be its limit superior.*

(a) *If $\alpha < x$, then there exists an $N \in \mathbf{N}$ such that $u_n < x$ for every $n \geq N$.*

(b) *If $x < \alpha$ and $N \in \mathbf{N}$, then there exists an $n \geq N$ such that $x < u_n$.*

Furthermore, these properties characterize the limit superior completely, in the sense that it is the only extended real number satisfying these properties.

Of course, an analogous result holds for the limit inferior.

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Consider an extended real sequence $(u_n)_{n=0}^{\infty}$. We say that an extended real number u is a *subsequential limit* of $(u_n)_{n=0}^{\infty}$ if there exists a subsequence of $(u_n)_{n=0}^{\infty}$ converging to u.

Proposition 6. *Suppose that u is any subsequential limit of the extended real sequence $(u_n)_{n=0}^{\infty}$. Then one has*

$$\liminf_{n\to\infty} u_n \le u \le \limsup_{n\to\infty} u_n.$$

In other words, the limit inferior and the limit superior of a sequence provide bounds on the subsequential limits of said sequence.

Proposition 7. *Let $(u_n)_{n=0}^{\infty}$ be a sequence of extended real numbers. Then both its limit inferior and limit superior are subsequential limits of $(u_n)_{n=0}^{\infty}$.*

Corollary 8. *Let U be the set of all subsequential limits of the extended real sequence $(u_n)_{n=0}^{\infty}$. Then U is closed, and one has*

$$\liminf_{n\to\infty} u_n = \inf U \quad \text{and} \quad \limsup_{n\to\infty} u_n = \sup U.$$

In light of the last two results, we see that *the limit inferior and the limit superior of a sequence is the minimal and maximal subsequential limit of that sequence.*

Proposition 9. *A sequence $(u_n)_{n=0}^{\infty}$ of extended real numbers is convergent if and only if its limit inferior and limit superior are equal, in which case one has*

$$\lim_{n\to\infty} u_n = \liminf_{n\to\infty} u_n = \limsup_{n\to\infty} u_n.$$