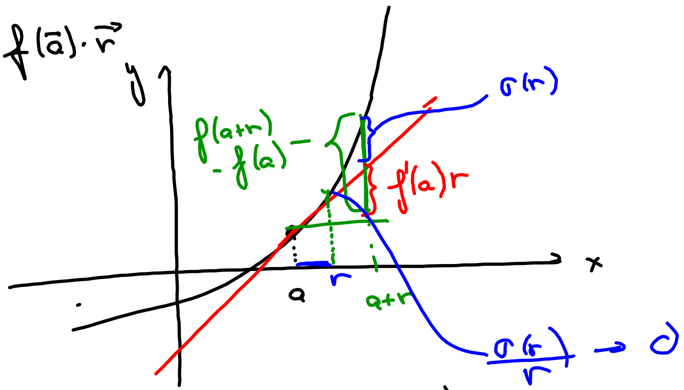
Deriverbarhet

deriverbar

Forrige gang: For "snille" funksjoner så er

$$f'(\vec{a};\vec{r}) = \nabla f(\vec{a})\cdot\vec{r}$$

Utgangspunkt.



$$\sigma(r) = f(a+r) - f(a) - f'(a)r \quad \text{vil} \quad \frac{\sigma(r)}{|r|} \to 0$$

I høyere dimensjoner:

$$\sigma(\vec{r}) = f(\vec{a}+\vec{r}) - f(\vec{a}) - \nabla f(\vec{a})\cdot\vec{r} \qquad \frac{\sigma(\vec{r})}{|\vec{r}|} \to 0$$

Definisjon: En funksjon $f: \mathbb{R}^n \to \mathbb{R}$ er deriverbar i $\vec{a}$ dersom alle de partiellderiverte i $\vec{a}$ eksisterer og

$$\sigma(\vec{r}) = f(\vec{a}+\vec{r}) - f(\vec{a}) - \nabla f(\vec{a})\cdot\vec{r}$$

går mot 0 raskere enn $\vec{r}$, dvs $\lim_{\vec{r}\to 0} \frac{\sigma(\vec{r})}{|\vec{r}|} = 0$.

Teorem: Hvis f er deriverbar i $\vec{a}$, så er

$$f'(\vec{a};\vec{r}) = \nabla f(\vec{a})\cdot\vec{r}$$

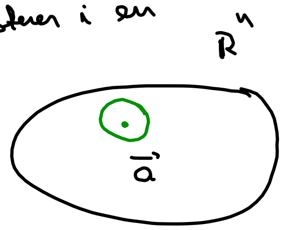
Bevis: $f'(\vec{a},\vec{r}) = \lim_{h\to 0} \frac{f(\vec{a}+h\vec{r}) - f(a)}{h}$

$$= \lim_{h\to 0} \frac{\nabla f(\vec{a})\cdot(\cancel{h}\vec{r}) + \sigma(h\vec{r})}{\cancel{h}}$$

$$= \lim_{h\to 0} \nabla f(\vec{a})\cdot\vec{r} + \lim_{h\to 0} \underbrace{\frac{\sigma(h\vec{r})}{h|\vec{r}|}}_{\to 0}\cdot|\vec{r}|$$

$$= \nabla f(\vec{a})\cdot\vec{r}$$

Teorem: Hvis de partiellderiverte til f eksisterer i en omegn om $\vec{a}$ og de er kontinuerlige i $\vec{a}$, så er f deriverbar i $\vec{a}$.



Eksempel: Finn $f'(\vec{a};\vec{r})$ når

$$f(x,y) = x^2 + 2xy^2$$

og $\vec{a} = (2,1)$, $\vec{r} = (-3,1)$.

Regner ut gradienten:

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= 2x + 2y^2 \\ \frac{\partial f}{\partial y} &= 2x2y = 4xy \end{aligned}\right\} \text{kontinuerlig} \Downarrow \text{ f deriverbar}$$

$$\nabla f = (2x+2y^2, 4xy)$$

Siden f er deriverbar, så er

$$f'(\vec{a};\vec{r}) = \nabla f(\vec{a}) \cdot \vec{r}$$

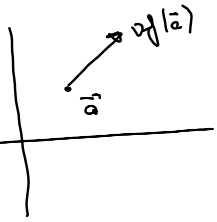
$$= (2\cdot 2 + 2\cdot 1^2, 4\cdot 2\cdot 1)\cdot(-3,1)$$

$$= (6,8)\cdot(-3,1) = -18+8 = \underline{\underline{-10}}$$

Oppsummering: Hvis de partiellderiverte er kontinuerlige, så er f deriverbar og $f'(\vec{a};\vec{r}) = \nabla f(\vec{a})\cdot\vec{r}$.

Geometrisk tolkning av gradienten.

Anta at f er deriverbar



Setning: Funksjonen stiger raskest i $\vec{a}$ i den retningen som $\nabla f(\vec{a})$ peker og stigningen er $|\nabla f(\vec{a})|$

Bevis: La $\vec{u}$ være en enhetsvektor, dvs $|\vec{u}|=1$. Da er

$$f'(\vec{a};\vec{u}) = \nabla f(\vec{a})\cdot\vec{u} \leq |\nabla f(\vec{a})|\,\underbrace{|\vec{u}|}_{1} = |\nabla f(\vec{a})|$$

Schwarz ulikhet: $|\vec{x}\cdot\vec{y}| \leq |\vec{x}||\vec{y}|$

dvs stigningen er alltid mindre enn eller lik $|\nabla f(\vec{a})|$. Men hvis $\vec{u} = \frac{\nabla f(\vec{a})}{|\nabla f(\vec{a})|}$, så

$$f'(\vec{a};\vec{u}) = \nabla f(\vec{a})\cdot\frac{\nabla f(\vec{a})}{|\nabla f(\vec{a})|} = \frac{1}{|\nabla f(\vec{a})|}\left(\nabla f(\vec{a})\cdot\nabla f(\vec{a})\right)$$

$$= \frac{1}{\cancel{|\nabla f(\vec{a})|}}|\nabla f(\vec{a})|^{\cancel{2}} = \underline{\underline{|\nabla f(\vec{a})|}}$$

Eksempel: I hvilken retning vokser $f(x,y,z) = xy^2z + x$ raskest i punktet $(1,-1,2)$?

Svar: I gradientens retning.

Finner gradienten

$$\nabla f(x,y,z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (y^2z+1, 2xyz, xy^2)$$

$$\nabla f(1,-1,2) = ((-1)^2\cdot 2+1, 2\cdot 1\cdot(-1)\cdot 2, 1\cdot(-1)^2)$$

$$= \underline{(3,-4,1)} \quad \underline{\text{raskest stigning}}.$$

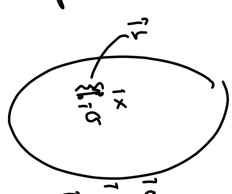
Teorem: Dersom f er deriverbar i $\vec{a}$, så er f kontinuerlig i $\vec{a}$.

Bevis:

$$\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = \lim_{\vec{r}\to\vec{0}} f(\vec{a}+\vec{r})$$

$$= \lim_{\vec{r}\to\vec{0}}\left[f(\vec{a}) + \nabla f(\vec{a})\cdot\vec{r} + \sigma(\vec{r})\right]$$

$$= f(\vec{a}) + 0 + 0 = \underline{f(\vec{a})}$$



$\vec{r} = \vec{x} - \vec{a}$

$\vec{x} = \vec{a} + \vec{r}$

Høyere ordens deriverte

$f(x,y)$ to partiellderiverte $\underline{\frac{\partial f}{\partial x}(x,y)}$ $\underline{\frac{\partial f}{\partial y}(x,y)}$

Annenderivate:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}(x,y)\right) = \frac{\partial^2 f}{\partial x^2}(x,y)$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}(x,y)\right) = \frac{\partial^2 f}{\partial y \partial x}(x,y)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}(x,y)\right) = \frac{\partial^2 f}{\partial x \partial y}(x,y)$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}(x,y)\right) = \frac{\partial^2 f}{\partial y^2}(x,y)$$

Eksempel: $f(x,y) = x^2 y + 2xe^y$

$\frac{\partial f}{\partial x} = \underline{2xy + 2e^y}$ $\frac{\partial f}{\partial y} = \underline{x^2 + 2xe^y}$

Annenordens partiellderivate:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(2xy + 2e^y) = 2y$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(2xy + 2e^y) = \underline{2x + 2e^y}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(x^2 + 2xe^y) = \underline{2x + 2e^y}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(x^2 + 2xe^y) = 2xe^y$$

Teorem: Anta at $f(x_1, \dots, x_n)$ er en funksjon av n variable og at $\frac{\partial^2 f}{\partial x_j \partial x_i}$ og $\frac{\partial^2 f}{\partial x_i \partial x_j}$ er definert i et område rundt $\vec{a}$ og kontinuerlige i $\vec{a}$. Da er

$$\frac{\partial^2 f}{\partial x_j \partial x_i}(\vec{a}) = \frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{a}).$$

Slagord: De blandede partiellderiverte er like!

Høyere ordens deriverte:

$$\frac{\partial^3 f}{\partial x \partial y \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right)$$

Derivete av funksjoner $\vec{F}:\mathbb{R}^n \to \mathbb{R}^m$

Anta $\vec{F}(x_1,..,x_n) = (F_1(x_1,..,x_n),\dots,F_m(x_1,..,x_n))$

Jacobimatrisen til $\vec{F}$:

$$\vec{F}'(\vec{a}) = \begin{pmatrix} \frac{\partial F_1}{\partial x_1}(\vec{a}) & \frac{\partial F_1}{\partial x_2}(\vec{a}) & \cdots & \frac{\partial F_1}{\partial x_n}(\vec{a}) \\ \frac{\partial F_2}{\partial x_1}(\vec{a}) & \frac{\partial F_2}{\partial x_2}(\vec{a}) & \cdots & \frac{\partial F_2}{\partial x_n}(\vec{a}) \\ \vdots & \vdots & & \vdots \\ \frac{\partial F_m}{\partial x_1}(\vec{a}) & \frac{\partial F_m}{\partial x_2}(\vec{a}) & \cdots & \frac{\partial F_m}{\partial x_n}(\vec{a}) \end{pmatrix} = \begin{pmatrix} \nabla F_1(\vec{a}) \\ \nabla F_2(\vec{a}) \\ \vdots \\ \nabla F_m(\vec{a}) \end{pmatrix}$$

Eksempel: $\vec{F}(x,y,z) = (\underbrace{xyz}_{F_1}, \underbrace{x^2y+z}_{F_2})$

$$\vec{F}'(x,y,z) = \begin{pmatrix} yz & xz & xy \\ 2xy & x^2 & 1 \end{pmatrix}$$

Definisjon: Anta at alle de partiell deriverte til $\vec{F}:\mathbb{R}^n \to \mathbb{R}^m$ eksisterer. Vi sier at $\vec{F}$ er deriverbar i $\vec{a}$ dersom

$$\vec{\sigma}(\vec{r}) = \vec{F}(\vec{a}+\vec{r}) - \vec{F}(\vec{a}) - \vec{F}'(\vec{a})\vec{r}$$

går mot 0 raskere enn $\vec{r}$, dvs $\lim_{\vec{r}\to 0} \frac{|\vec{\sigma}(\vec{r})|}{|\vec{r}|} = 0$.

Setning: $\vec{F}$ er deriverbar i $\vec{a}$ hvis og bare hvis hver koordinatfunksjon $F_1, F_2, .., F_m$ er kontinuerlig i $\vec{a}$

Teorem: Anta at A er en matrise slik at

$\vec{\sigma}(\vec{r}) = \underline{\vec{F}(\vec{a}+\vec{r}) - \vec{F}(\vec{a}) - A\vec{r}}$ går mot null raskere enn $\vec{r}$ (dvs $\lim \frac{|\vec{\sigma}(\vec{r})|}{|\vec{r}|} \to 0$),

da $\vec{\sigma}(\vec{r})$ går mot null raskere enn $\vec{r}$

så er $\vec{F}$ deriverbar i $\vec{a}$ og $A = \vec{F}'(\vec{a})$.