Repetisjon: Delbrøksoppspalting og komplekse tall

Søtt integral: $I = \int \sqrt{\tan x}\, dx$

Matematikkt uttrykk = aldeles forjævlig

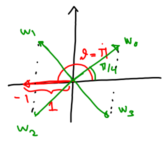
To depervasjonssubstitusjoner:
$u = \tan x$
$u = \sqrt{\tan x}$

$$u = \sqrt{\tan x} \Rightarrow u^2 = \tan x$$
$$x = \arctan u^2$$
$$dx = \frac{1}{1+(u^2)^2} 2u\, du = \frac{2u}{1+u^4}\, du$$

$$I = \int u \frac{2u}{1+u^4}\, du = \int \frac{2u^2}{u^4+1}\, du$$

Hurra, rasjonal funksjon. Må faktorisere u^4+1

Vi må løse ligningen $u^4+1=0$, dvs $\underbrace{u^4=-1}_{\text{fjerderøtter}}$ til -1.



Første fjerderoten:

$$w_0 = 1^{1/4} e^{i\pi/4} = \cos\frac{\pi}{4} + i \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$$

$$w_1 = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$$
$$w_2 = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$$
$$w_3 = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$$

$$u^4+1 = (u-w_0)(u-w_1)(u-w_2)(u-w_3)$$
$$= (u-w_0)(u-w_3)(u-w_1)(u-w_2)$$
$$= \left(u-\left(\tfrac{\sqrt{2}}{2}+i\tfrac{\sqrt{2}}{2}\right)\right)\left(u-\left(\tfrac{\sqrt{2}}{2}-i\tfrac{\sqrt{2}}{2}\right)\right)\left(u-\left(-\tfrac{\sqrt{2}}{2}+i\tfrac{\sqrt{2}}{2}\right)\right)\left(u-\left(-\tfrac{\sqrt{2}}{2}-i\tfrac{\sqrt{2}}{2}\right)\right)$$
$$= \underbrace{\left(\left(u-\tfrac{\sqrt{2}}{2}\right)+i\tfrac{\sqrt{2}}{2}\right)\left(\left(u-\tfrac{\sqrt{2}}{2}\right)-i\tfrac{\sqrt{2}}{2}\right)}\left(\left(u+\tfrac{\sqrt{2}}{2}\right)-i\tfrac{\sqrt{2}}{2}\right)\left(\left(u+\tfrac{\sqrt{2}}{2}\right)+i\tfrac{\sqrt{2}}{2}\right)$$
$$= \left[\left(u-\tfrac{\sqrt{2}}{2}\right)^2-\left(i\tfrac{\sqrt{2}}{2}\right)^2\right]\cdot\left[\left(u+\tfrac{\sqrt{2}}{2}\right)^2-\left(i\tfrac{\sqrt{2}}{2}\right)^2\right]$$
$$= \left[u^2-\sqrt{2}u+\tfrac{1}{2}+\tfrac{1}{2}\right]\left[u^2+\sqrt{2}u+\tfrac{1}{2}+\tfrac{1}{2}\right]$$
$$= (u^2-\sqrt{2}u+1)(u^2+\sqrt{2}u+1)$$

Tilbake til integralet:

$$I = \int \frac{2u^2}{u^4+1}\, du = \int \frac{2u^2}{(u^2-\sqrt{2}u+1)(u^2+\sqrt{2}u+1)}\, du$$

Delbrøksoppspalting:

$$\frac{2u^2}{(u^2-\sqrt{2}u+1)(u^2+\sqrt{2}u+1)} = \frac{Au+B}{u^2-\sqrt{2}u+1} + \frac{Cu+D}{u^2+\sqrt{2}u+1}$$

Ganger med $(u^2-\sqrt{2}u+1)(u^2+\sqrt{2}u+1)$:

$$2u^2 = (Au+B)(u^2+\sqrt{2}u+1) + (Cu+D)(u^2-\sqrt{2}u+1)$$
$$= Au^3+\sqrt{2}Au^2+Au+Bu^2+\sqrt{2}Bu+B$$
$$= Cu^3-\sqrt{2}Cu^2+Cu+Du^2-\sqrt{2}Du+D$$
$$= (A+C)u^3 + (\sqrt{2}A+B-\sqrt{2}C+D)u^2$$
$$(A+\sqrt{2}B+Cu-\sqrt{2}D)u + B+D$$

$A+C=0$, $\sqrt{2}A+B-\sqrt{2}C+D=2$
$A+\sqrt{2}B+C-\sqrt{2}D=0$, $B+D=0$

Sett $C=-A$ og $D=-B$ inn i de to midterste og løs:

Får: $C=-\frac{1}{\sqrt{2}}$, $D=0$, $A=\frac{1}{\sqrt{2}}$, $B=0$

Dermed $\dfrac{2u^2}{(u^2-\sqrt{2}u+1)(u^2+\sqrt{2}u+1)} = \underbrace{\dfrac{\frac{1}{\sqrt{2}}u}{u^2-\sqrt{2}u+1} - \dfrac{\frac{1}{\sqrt{2}}u}{u^2+\sqrt{2}u+1}}$

Regner ut: $I_2 = \int \frac{u}{u^2-\sqrt{2}u+1}\, du$ $\quad N = u^2-\sqrt{2}u+1$, $N' = 2u-\sqrt{2}$

$$= \frac{1}{2}\int \frac{2u}{u^2-\sqrt{2}u+1}\, du = \frac{1}{2}\int \frac{2u-\sqrt{2}+\sqrt{2}}{u^2-\sqrt{2}u+1}\, du$$

$$= \frac{1}{2}\int \frac{2u-\sqrt{2}}{u^2-\sqrt{2}u+1}\, du + \frac{\sqrt{2}}{2}\int \frac{1}{u^2-\sqrt{2}u+1}\, du$$

Ny variabel
$z = u^2-\sqrt{2}u+1$
$dz = (2u-\sqrt{2})\, du$

$$= \frac{1}{2}\ln|z| + \frac{\sqrt{2}}{2}\int \frac{1}{u^2-\sqrt{2}u+1}\, du$$
$$= \frac{1}{2}\ln(u^2-\sqrt{2}u+1) + \frac{\sqrt{2}}{2}\int \frac{1}{u^2-\sqrt{2}u+1}\, du$$

Må løse: $I_3 = \int \frac{1}{u^2-\sqrt{2}u+1}\, du$

$$= \int \frac{1}{\underbrace{u^2-\sqrt{2}u+\left(\frac{\sqrt{2}}{2}\right)^2}_{\left(u-\frac{\sqrt{2}}{2}\right)^2}\underbrace{-\left(\frac{\sqrt{2}}{2}\right)^2}_{-\frac{1}{2}}+1}\, du$$

$$= \int \frac{1}{\left(u-\frac{\sqrt{2}}{2}\right)^2+\frac{1}{2}}\, du = \int \frac{1}{\frac{1}{2}\left[2\left(u-\frac{\sqrt{2}}{2}\right)^2+1\right]}\, du$$

$$= \int \frac{2}{(\sqrt{2}u-1)^2+1}\, du$$

$w = \sqrt{2}u-1$
$dw = \sqrt{2}\, du$
$du = \frac{1}{\sqrt{2}}\, dw$

$$= \int \frac{\frac{2}{\sqrt{2}}\, dw}{w^2+1} = \sqrt{2}\arctan w + C$$
$$= \sqrt{2}\arctan(\sqrt{2}u-1) + C$$

$u = \sqrt{\tan x}$

Sluttsvar:

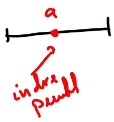
$$\int \sqrt{\tan x}\, dx = \frac{1}{2\sqrt{2}}\Big[\log(u^2-2\sqrt{u}+1) - \log(u^2+2\sqrt{u}+1) + 2\arctan(\sqrt{2}u-1) + 2\arctan(\sqrt{2}u+1)\Big] + C$$

der $u = \sqrt{\tan x}$

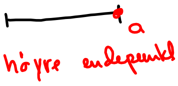
Kontinuitet og deriverbarhet

Definisjon: En funksjon $f: D_f \to \mathbb{R}$ er kontinuerlig i $a \in D_f$ hvis det for hver $\varepsilon > 0$ finnes en $\delta > 0$ slik at når $x \in D_f$ og $|x-a| < \delta$, så er $|f(x) - f(a)| < \varepsilon$.

Ved hjelp av grenseverdier:



f er kontinuerlig i a dersom $\lim\limits_{x\to a} f(x) = f(a)$



—— " —— dersom $\lim\limits_{x\to a^-} f(x) = f(a)$

—— " —— dersom $\lim\limits_{x\to a^+} f(x) = f(a)$

Eksempel:

$$f(x) = \begin{cases} \dfrac{e^x - 1}{x} & \text{for } x > 0 \\ \cos x & \text{for } x \le 0 \end{cases}$$

Er f kontinuerlig i 0? Må sjekke om

$$\lim_{x\to 0} f(x) = f(0) = \cos 0 = 1$$

Sjekker om $\lim\limits_{x\to 0^+} f(x) = 1$ og om $\lim\limits_{x\to 0^-} f(x) = 1$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{e^x - 1}{x} \overset{L'H}{=} \lim_{x\to 0^+} \frac{e^x}{1} = \frac{e^0}{1} = 1$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \cos x = \cos 0 = 1$$

Dermed er $\lim\limits_{x\to 0} f(x) = 1 = f(0)$, så f er kontinuerlig i 0.

Eksempel:

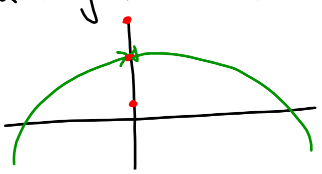
$$f(x) = \begin{cases} \dfrac{\ln(x+1)}{x} & \text{hvis } x > -1 \text{ og } x \neq 0, \\ a & \text{hvis } x = 0 \end{cases}$$

Finnes det en a slik at f blir kontinuerlig?

Regner ut

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\ln(x+1)}{x} \overset{L'H}{=} \lim_{x\to 0} \frac{\frac{1}{x+1}}{1} = 1.$$

Setter vi $f(0) = a = 1$, får vi at $\lim\limits_{x\to 0} f(x) = f(0)$



Deriverbarhet:

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$ f deriverbar hvis denne grensen finnes.

Eksempel:

$$f(x) = \begin{cases} \frac{e^x-1}{x} & \text{for } x>0 \\ \cos x & \text{for } x \le 0 \end{cases}$$

Er denne deriverbar i 0? Må sjekke om

$$\lim_{x\to 0} \frac{f(x)-f(0)}{x-0}$$ eksisterer.

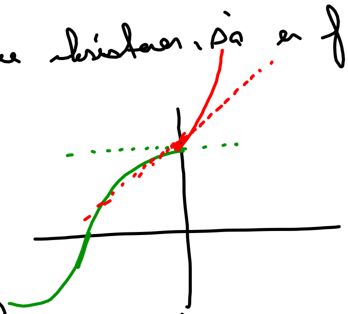
Sjekker ensidige grenser:

$$\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} \frac{\frac{e^x-1}{x}-1}{x}$$

$$= \lim_{x\to 0^+} \frac{e^x-1-x}{x^2} \overset{L'H}{=} \lim_{x\to 0^+} \frac{e^x-1}{2x} \overset{L'H}{=} \lim_{x\to 0^+} \frac{e^x}{2} = \frac{1}{2}$$

$$\lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \frac{\cos x - 1}{x-0} = \lim_{x\to 0^-} \underbrace{\frac{\cos x - \cos 0}{x-0}}_{=\sin(0)} = 0$$

Siden $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0}$ ikke eksisterer, så er f ikke deriverbar i 0.



Eksempel:

$$f(x) = \begin{cases} \frac{\ln(x+1)}{x} & \text{når } x \neq 0 \\ 1 & \text{når } x=0 \end{cases}$$

Er f deriverbar i $x=0$?

$$\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{\frac{\ln(x+1)}{x} - 1}{x-0}$$

$$= \lim_{x\to 0} \frac{\ln(x+1)-x}{x^2} \overset{L'H}{=} \lim_{x\to 0} \frac{\frac{1}{x+1}-1}{2x}$$

$$\overset{L'H}{=} \lim_{x\to 0} \frac{-\frac{1}{(x+1)^2}}{2} = -\frac{1}{2}$$

Så, $f(x)$ er deriverbar i $x=0$ med $f'(0) = -\frac{1}{2}$