Grenseverdier

Mange typer:

$\lim_{n\to\infty} a_n$

$\lim_{\substack{x\to\infty \\ x\to-\infty}} f(x) = \begin{cases} a \\ \infty \end{cases}$

$\lim_{x\to a} f(x)$

$\lim_{\substack{x\to a^+ \\ x\to a^-}} f(x)$

Noen triks: Trekk ut "høyest faktor"

$$\lim_{n\to\infty} \frac{8n^3+7n^2+4}{7n^3+2n+5} = \lim_{n\to\infty} \frac{\cancel{n^3}(8+\overbrace{\tfrac{7}{n}+\tfrac{4}{n^3}}^{0})}{\cancel{n^3}(7+\underbrace{\tfrac{2}{n^2}+\tfrac{5}{n^3}}_{0})} = \underline{\underline{\frac{8}{7}}}$$

Gange med den konjugerte:

$$\lim_{x\to\infty}\left(\sqrt{x^2+2x}-x\right) = \lim_{x\to\infty} \frac{(\sqrt{x^2+2x}-x)(\sqrt{x^2+2x}+x)}{\sqrt{x^2+2x}+x}$$

$$= \lim_{x\to\infty} \frac{(\sqrt{x^2+2x})^2-x^2}{\sqrt{x^2+2x}+x} = \lim_{x\to\infty} \frac{\cancel{x^2}+2x-\cancel{x^2}}{\sqrt{x^2+2x}+x}$$

$$= \lim_{x\to\infty} \frac{2x}{\sqrt{x^2(1+\frac{2}{x})}+x} = \lim_{x\to\infty} \frac{2x}{\underbrace{\sqrt{x^2}}_{x}\sqrt{1+\frac{2}{x}}+x}$$

$\sqrt{ab} = \sqrt{a}\sqrt{b}$

$$= \lim_{x\to\infty} \frac{2\cancel{x}}{\cancel{x}(\sqrt{1+\frac{2}{x}}+1)} = \lim_{x\to\infty} \frac{2}{\sqrt{1+\frac{2}{x}}+1} = \frac{2}{2} = \underline{\underline{1}}$$

L'Hôpitals regel: Anta $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \begin{cases} 0 \\ \infty \end{cases}$ og

$\lim_{x\to a} \frac{f'(x)}{g'(x)} = L.$ Da er

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)} = L.$$

Eksempel:

$$\lim_{x\to 0} \frac{\arctan x - x}{x^3} \overset{L'H}{=} \lim_{x\to 0} \frac{\left(\frac{1}{1+x^2}-1\right)(1+x^2)}{3x^2(1+x^2)}$$

$$= \lim_{x\to 0}\frac{1-(1+x^2)}{3x^2(1+x^2)} = \lim_{x\to 0}\frac{-\cancel{x^2}}{3\cancel{x^2}(1+x^2)} = \lim_{x\to 0}\frac{-1}{3(\underbrace{1+x^2}_{\to 0})} = \underline{\underline{-\frac{1}{3}}}$$

Eksempel:

$$\lim_{x\to\infty}\left(\sqrt{x^2+2x}-x\right) = \lim_{x\to\infty} \overset{\infty}{x}\left(\sqrt{1+\tfrac{2}{x}}-1\right)$$

$$= \lim_{x\to\infty} \frac{\sqrt{1+\frac{2}{x}}-1}{\frac{1}{x}} \overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{1}{\cancel{2}\sqrt{1+\frac{2}{x}}}\left(-\cancel{\frac{2}{x^2}}\right)}{\cancel{-\frac{1}{x^2}}}$$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{1+2/x}} = \frac{1}{1} = \underline{1}$$

Eksempel:

$$\lim_{x\to\infty}\left(\overbrace{1+\sin\tfrac{1}{x}}^{\to 1}\right)^{2x \to\infty}$$

$$\lim_{x\to\infty}\left(1+\sin\tfrac{1}{x}\right)^{2x} = \lim_{x\to\infty}\left[e^{\ln(1+\sin\frac{1}{x})}\right]^{2x}$$

$$= \lim_{x\to\infty} e^{\overset{\infty}{2x}\,\overset{0}{\ln(1+\sin\frac{1}{x})}} = \underline{\underline{e^2}}$$

Mellomregning:

$$\lim_{x\to\infty} 2x\ln\left(1+\sin\tfrac{1}{x}\right) = \lim_{x\to\infty}\frac{2\ln(1+\sin\frac{1}{x})}{\frac{1}{x}}$$

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{2\,\frac{1}{1+\sin\frac{1}{x}}\cos\frac{1}{x}\cdot\cancel{\left(-\frac{1}{x^2}\right)}}{\cancel{-\frac{1}{x^2}}} = \lim_{x\to\infty}\frac{2\cos\frac{1}{x}}{1+\underbrace{\sin\tfrac{1}{x}}_{0}} = \frac{2\cdot 1}{1+0} = \underline{2}$$

Bruk av grenser: kontinuitet, deriverbarhet, asymptoter.

Eksempel: Undersøk om $f(x) = \sqrt[3]{8x^3+x^2}$ har en asymptote når $x \to \infty$.

$y = ax+b$

slik $\lim_{x\to\infty} [f(x) - (ax+b)] = 0$

Metode

$\lim_{x\to\infty} \frac{f(x)}{x} = a$

$\lim_{x\to\infty} [f(x) - ax] = b$

(i) $\lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{\sqrt[3]{8x^3+x^2}}{x} = \lim \frac{\cancel{x}\sqrt[3]{8+\frac{1}{x}}}{\cancel{x}}$

$= \lim_{x\to\infty} \sqrt[3]{8+\frac{1}{x}} = \sqrt[3]{8} = \underline{\underline{2}}$

(ii) $\lim_{x\to\infty} [f(x) - 2x] = \lim_{x\to\infty} \left(\sqrt[3]{\underbrace{8x^3}_{(2x)^3}+x^2} - 2x\right)$

$= \lim_{x\to\infty} \left(2x\sqrt[3]{1+\frac{1}{8x}} - 2x\right) = \lim_{x\to\infty} 2x\left(\sqrt[3]{1+\frac{1}{8x}} - 1\right)$

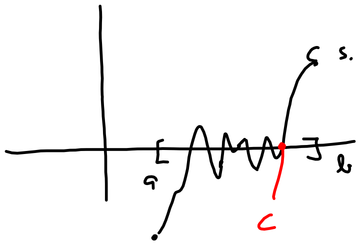
$= \lim_{x\to\infty} 2 \frac{\left(1+\frac{1}{8x}\right)^{1/3} - 1}{\frac{1}{x}} \overset{L'H}{=} 2 \lim_{x\to\infty} \frac{\frac{1}{3}\left(1+\frac{1}{8x}\right)^{-2/3}\frac{1}{8}\left(\cancel{-\frac{1}{x^2}}\right)}{\cancel{-\frac{1}{x^2}}}$

$= \frac{1}{12} \lim_{x\to\infty} \left(1+\frac{1}{8x}\right)^{-2/3} = \underline{\underline{\frac{1}{12}}}$

Asymptote: $y = 2x + \frac{1}{12}$

Tre store teoremer

Skjæringssetningen: Hvis $f:[a,b]\to\mathbb{R}$ er kontinuerlig og $f(a), f(b)$ har motsatte fortegn, så finnes det en $c\in(a,b)$ slik at $f(c)=0$



Eksempel: Vis $\sqrt[17]{2}$ finnes, dvs et tall c s.a. $c^{17}=2$.

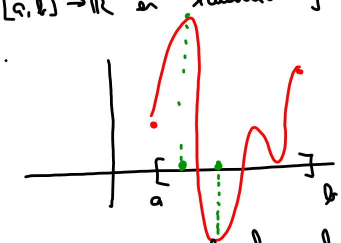
La $f(x)=x^{17}-2$, da

$$f(0)=0^{17}-2=-2<0$$
$$f(2)=2^{17}-2>0$$

Ifølge skjæringssetningen finnes det en c slik

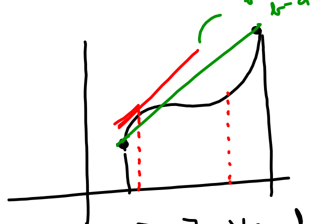
$$c^{17}-2=f(c)=0 \text{, dvs}$$
$$c^{17}=2.$$

Ekstremverdisetningen: Hvis $f:[a,b]\to\mathbb{R}$ er kontinuerlig, så har den maks- og minpunkter.



Middelverdisetningen: Hvis $f:[a,b]\to\mathbb{R}$ er kontinuerlig i hele $[a,b]$ og deriverbar i alle $x\in(a,b)$, så finnes det en $c\in(a,b)$ slik at

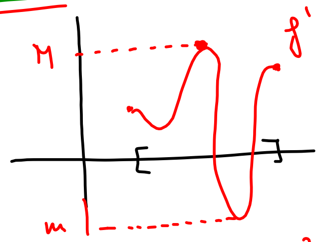
$$f'(c)=\frac{f(b)-f(a)}{b-a}$$



Eksempel: Anta at f' er kontinuerlig på $[a,b]$. Vis at det finnes en konstant K slik at

$$|f(x)-f(y)|\leq K|x-y| \quad \text{for alle } x,y\in[a,b]$$

Siden f' er kontinuerlig, så er f' begrenset, dvs det finnes en konstant K slik $|f'(x)|\leq K$ for alle $x\in[a,b]$.



$K=\max\{|M|,|m|\}$

Ved middelverdisetningen

$$\frac{|f(x)-f(y)|}{|x-y|}=|\cancel{f'(c)}|\leq K$$

Ganger med $|x-y|$:

$$|f(x)-f(y)|\leq K|x-y|$$

Eksempel: Anta at $f'(0)=f(0)$ og $f'(1)=f(1)$ og at $f''(x)$ eksisterer for alle $x\in(0,1)$. Vis at det finnes en $c\in(0,1)$ der $f''(c)=f'(c)$.

Bruker middelverdisetningen på $h(x)=f(x)-f'(x)$.

Da $h(0)=f(0)-f'(0)=0$, $h(1)=f(1)-f'(1)=0$

Ifølge middelverdisetningen finnes det $c\in(0,1)$ der

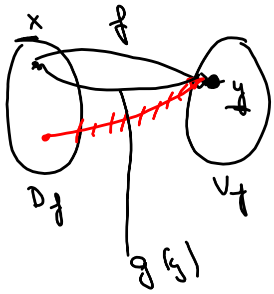
$$h'(c)=\frac{h(1)-h(0)}{1-0}=0.$$

Men $h'(x)=f'(x)-f''(x)$, så

$$f'(c)-f''(c)=h'(c)=0,\ \text{dvs}\ f'(c)=f''(c).$$

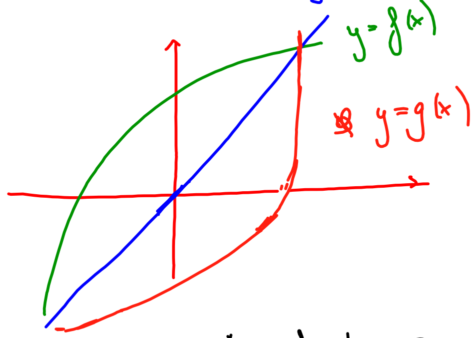
Omvendte funksjoner

$f: D_f \to V_f$ er injektiv dersom det til hver $y \in V_f$ bare finnes 'én x slik at $f(x)=y$.

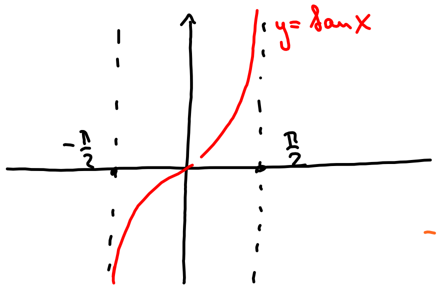


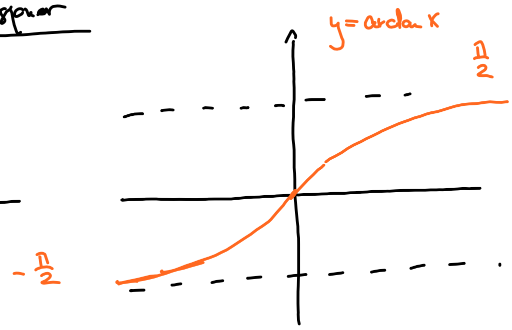
Vi definerer den omvendte funksjonen $g: V_f \to D_f$ ved at $g(y)=x$, der x er det ene elementet slik $f(x)=y$

Derivasjon: $g'(y) = \dfrac{1}{f'(x)}$ der $y=f(x)$ evt. $x=g(y)$ forutsatt at $f'(x) \neq 0$.

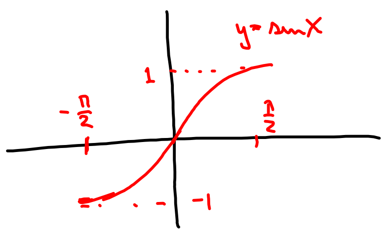


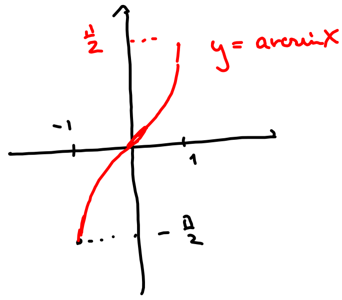
Arcusfunksjoner





$$(\arctan x)' = \frac{1}{1+x^2}$$





$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$$

Eksempel: $\displaystyle\int \frac{x}{\sqrt{1-\underbrace{x^4}_{u^2}}}\,dx$

$u = x^2$
$du = 2x\,dx$
$x\,dx = \frac{1}{2}\,du$

$$= \int \frac{\frac{1}{2}\,du}{\sqrt{1-u^2}} = \frac{1}{2}\arcsin u + C = \frac{1}{2}\arcsin x^2 + C$$