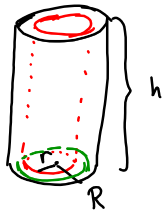
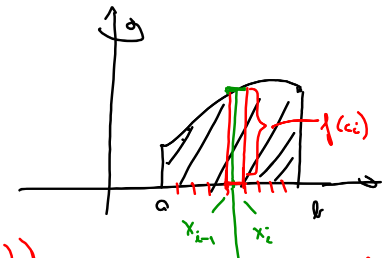
Omdreiningslegemer om y-aksen

Volumet til et rør:



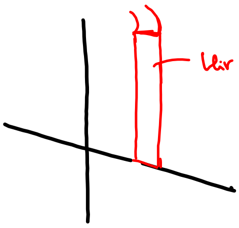
$$V = \pi R^2 h - \pi r^2 h = \pi h (R^2 - r^2)$$

$$= \pi h (R+r)(R-r) = \underbrace{2\pi h}_{\text{høyden}} \underbrace{\tfrac{R+r}{2}}_{\text{middels radius}} \underbrace{(R-r)}_{\text{tykkelsen}}$$



dreier om y-aksen

c_i = midtpunktet mellom x_{i-1} og x_i



Blir et rør med $h = f(c_i)$, tykkelsen er $(x_i - x_{i-1})$, middels radius c_i

$$V_i = 2\pi f(c_i)\, c_i\, (x_i - x_{i-1})$$

$$V \approx \sum_{i=1}^{n} \underbrace{2\pi f(c_i) c_i}(x_i - x_{i-1})$$

Riemannsum til funksjon $2\pi f(x)x$.

$$\downarrow$$

$$\int_a^b 2\pi f(x) x\, dx = 2\pi \int_a^b x f(x)\, dx$$

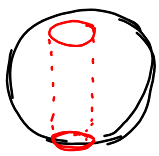
Riemannsum
$$\sum f(c_i)(x_i - x_{i-1})$$

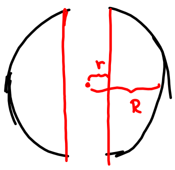
Konklusjon: Hvis vi roterer en positiv funksjon $f(x)$, $0 \le a \le x \le b$ rundt y-aksen, så blir omdreiningsvolumet lik

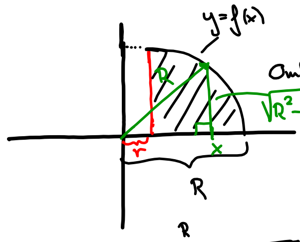
$$V = 2\pi \int_a^b x f(x)\, dx.$$

Eksempel: Kule med hull med sentrum.

Hva er volumet?







Omdreiningslegemet gir en halv volumet.

$$V = 2 \cdot 2\pi \int_r^R x f(x)\, dx = 4\pi \int_r^R x\sqrt{R^2 - x^2}\, dx$$

Mellomregning: $\int x\sqrt{R^2 - x^2}\, dx$

$u = R^2 - x^2$
$du = -2x\, dx$
$x\, dx = -\frac{1}{2} du$

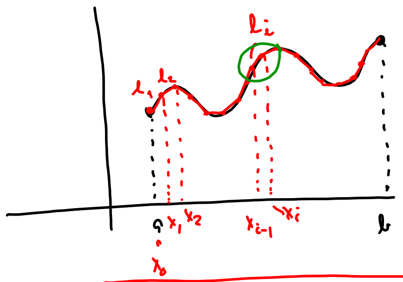
$$= \int -\tfrac{1}{2}\sqrt{u}\, du = -\tfrac{1}{2}\int u^{1/2}\, du$$

$$= -\tfrac{1}{2}\left[\frac{u^{3/2}}{3/2}\right] + C = -\tfrac{1}{2}\cdot\tfrac{2}{3} u^{3/2} + C = -\tfrac{1}{3}(R^2 - x^2)^{3/2} + C$$

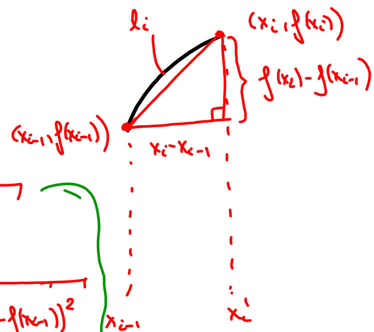
$$V = 4\pi \int_r^R x\sqrt{R^2 - x^2}\, dx = 4\pi \left[-\tfrac{1}{3}(R^2 - x^2)^{3/2}\right]_r^R$$

$$= 0 + \frac{4\pi}{3}(R^2 - r^2)^{3/2} = \frac{4\pi}{3}(R^2 - r^2)^{3/2}$$

Buelengde



Hvor lang er funksjonsgrafen?



$$l_i = \sqrt{(x_i - x_{i-1})^2 + (f(x_i)-f(x_{i-1}))^2}$$

$$l \approx \sum l_i = \sum_{i=1}^{n} \sqrt{(x_i-x_{i-1})^2 + (f(x_i)-f(x_{i-1}))^2}$$

$$= \sum_{i=1}^{n} \sqrt{1 + \left(\frac{f(x_i)-f(x_{i-1})}{x_i - x_{i-1}}\right)^2}\,(x_i - x_{i-1})$$

$\sum f(c_i)(x_i - x_{i-1})$

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

$$= \sum_{i=1}^{n} \sqrt{1 + f'(c_i)^2}\,(x_i - x_{i-1})$$

Riemannsum $\sqrt{1+f'(x)^2}$

$$\int_a^b \sqrt{1+f'(x)^2}\,dx$$

Konklusjon: Hvis f har en kontinuerlig derivert på $[a,b]$, så er lengden til grafen til f gitt ved

$$L = \int_a^b \sqrt{1+f'(x)^2}\,dx \qquad \text{(buelengden)}$$

Eksempel: $f(x) = \ln x - \frac{x^2}{8}$ og $1 \le x \le e$

Finn buelengden:

$$f'(x) = \frac{1}{x} - \frac{2x}{8} = \frac{1}{x} - \frac{x}{4}$$

Buelengden:

$$L = \int_1^e \sqrt{1 + \left(\frac{1}{x} - \frac{x}{4}\right)^2}\,dx = \int_1^e \sqrt{1 + \frac{1}{x^2} - 2\cdot\frac{1}{\cancel{x}}\cdot\frac{\cancel{x}}{4} + \frac{x^2}{16}}\,dx$$

$$= \int_1^e \sqrt{1 + \frac{1}{x^2} - \frac{1}{2} + \frac{x^2}{16}} = \int_1^e \sqrt{\frac{1}{x^2} + \frac{1}{2} + \frac{x^2}{16}}$$

$$= \int_1^e \sqrt{\left(\frac{1}{x} + \frac{x}{4}\right)^2}\,dx = \int_1^e \left(\frac{1}{x} + \frac{x}{4}\right)dx$$

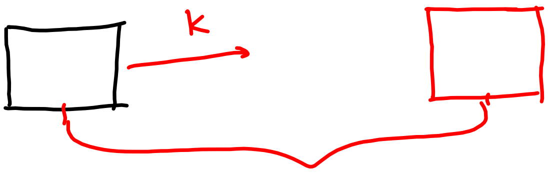
$$\frac{1}{x^2} + \underbrace{2\cdot\frac{1}{x}\cdot\frac{x}{4}}_{1/2} + \frac{x^2}{16}$$

$$= \left[\ln|x| + \frac{x^2}{8}\right]_1^e = \left(\underbrace{\ln e}_{1} + \frac{e^2}{8}\right) - \left(\underbrace{\ln 1}_{0} + \frac{1}{8}\right)$$

$$= 1 + \frac{e^2}{8} - \frac{1}{8} = \frac{7}{8} + \frac{e^2}{8} = \underline{\underline{\frac{e^2+7}{8}}}$$

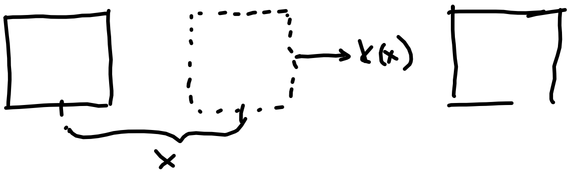
Kraft og arbeid

Naturfag: Arbeid = kraft gangen vei



$A = Ks$ konstant kraft

Hva hvis kraften varierer?



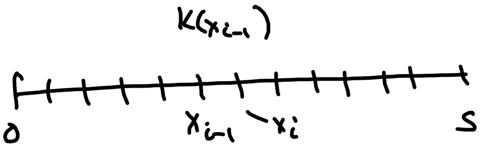
Hva blir arbeidet:

$$A_i = K(x_{i-1})(x_i - x_{i-1})$$

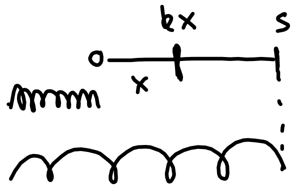
$$A = \sum K(x_{i-1})(x_i - x_{i-1})$$

$$\downarrow$$

$$\int_0^s K(x)\,dx$$



Eksempel



Hookes lov

$K(x) = kx$

$$A = \int_0^s kx\,dx = \left[k\frac{x^2}{2}\right]_0^s = \frac{k}{2}s^2$$

Delvis integrasjon (9.1)

$$\int u v' \, dx = uv - \int u' v \, dx$$

Bevis: $(uv)' = u'v + uv'$

Integrer

$$uv + C = \int u' v \, dx + \int u v' \, dx$$

$$\int u v' \, dx = uv - \int u' v \, dx + \cancel{C}$$

Eksempel: $\int x e^x \, dx = x e^x - \int 1 \cdot e^x \, dx$

$$= x e^x - \int e^x \, dx = x e^x - e^x + C$$

$u = x \quad v' = e^x$

$u' = 1 \quad v = e^x$

Eksempel: $\int x^2 \sin x \, dx$

$$= -x^2 \cos x - \int 2x(-\cos x) \, dx$$

$$= -x^2 \cos x + 2\int x \cos x \, dx$$

$$= -x^2 \cos x + 2\left[x \sin x - \int 1 \cdot \sin x \, dx\right]$$

$$= -x^2 \cos x + 2x \sin x - 2\int \sin x \, dx$$

$$= -x^2 \cos x + 2x \sin x + 2\cos x + C$$

$u = x^2 \quad v' = \sin x$

$u' = 2x \quad v = -\cos x$

$u = x \quad v' = \cos x$

$u' = 1 \quad v = \sin x$

Eksempel: $\int \ln x \, dx = \int 1 \cdot \ln x \, dx$

$$= x \ln x - \int \frac{1}{x} \cdot x \, dx = x \ln x - \int 1 \, dx$$

$$= x \ln x - x + C$$

$u = \ln x \quad v' = 1$

$u' = \frac{1}{x} \quad v = x$

Eksempel: $\int x \cdot \arctan x \, dx$

$$= \frac{x^2}{2} \arctan x - \int \frac{\frac{x^2}{2}}{1+x^2} \, dx$$

$$= \frac{x^2}{2} \arctan x - \frac{1}{2}\int \frac{x^2}{x^2+1} \, dx = \frac{x^2}{2} \arctan x - \frac{1}{2}\int \frac{(x^2+1)-1}{x^2+1} \, dx$$

$$= \frac{x^2}{2} \arctan x - \frac{1}{2}\int \left(1 - \frac{1}{x^2+1}\right) dx$$

$$= \frac{x^2}{2} \arctan x - \frac{1}{2}\left(x - \arctan x\right) + C$$

$$= \left(\frac{x^2}{2} + \frac{1}{2}\right) \arctan x - \frac{x}{2} + C = \frac{x^2+1}{2} \arctan x - \frac{x}{2} + C$$

$u = \arctan x \quad v' = x$

$u' = \frac{1}{1+x^2} \quad v = \frac{x^2}{2}$

Eksempel: $I = \int \sin^2 x \, dx$

$$= -\sin x \cos x - \int (-\cos^2 x) \, dx$$

$$= -\sin x \cos x + \int \cos^2 x \, dx$$

$$= -\sin x \cos x + \int (1 - \sin^2 x) \, dx$$

$$= -\sin x \cos x + \int 1 \, dx - I$$

dvs

$$I = -\sin x \cos x + x - I$$

$$2I = -\sin x \cos x + x$$

$$I = -\frac{1}{2}\sin x \cos x + \frac{x}{2} + C$$

$u = \sin x \quad v' = \sin x$

$u' = \cos x \quad v = -\cos x$