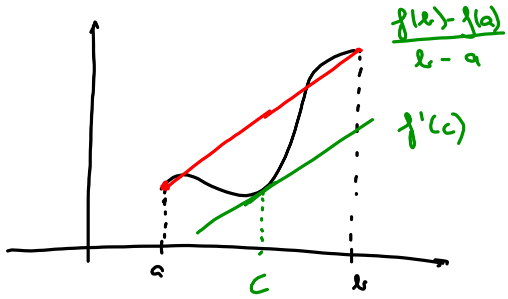
# Middelverdisetningen



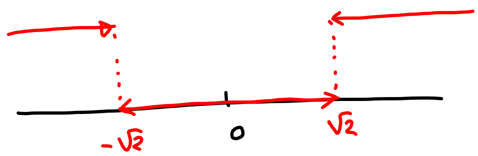

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Vet: Hvis $f(x)$ er konstant, så er $f'(x)=0$ for alle $x$

Men hvis $f'(x)=0$ for alle, må da $f$ være konstant.

Ikke helt trivielt: $f: \mathbb{Q} \to \mathbb{Q}$

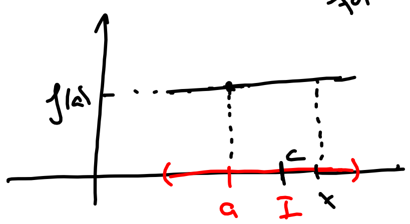
$$f(x) = \begin{cases} 0 & \text{hvis } x^2 < 2 \\ 1 & \text{ellers.} \end{cases}$$




$f'(x) = 0$ for alle $x \in \mathbb{Q}$.

Setning: Anta at $I$ er et intervall og at $f'(x)=0$ for alle $x \in I$. Da er $f$ konstant på $I$.

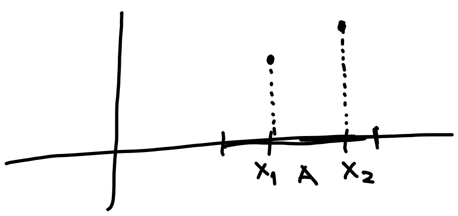
Bevis: Velg et punkt $a \in I$. Det holder å vise at $f(x)=f(a)$ for alle $x \in I$. Bruker MVS på intervallet $[a,x]$:



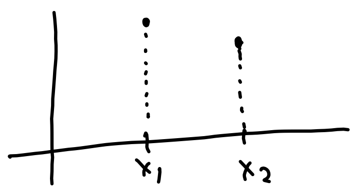

$$\frac{f(x)-f(a)}{x-a} = f'(c) = 0,$$

dvs $f(x)-f(a)=0$, altså $f(x)=f(a)$

Definisjonen: En funksjon $f$ er voksende på mengde $A$ dersom hver gang $x_1, x_2 \in A$ med $x_1 < x_2$, så er $f(x_1) \le f(x_2)$



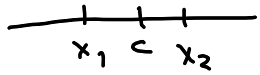

$f$ er avtagende på $A$ dersom vi istedet har $f(x_1) \ge f(x_2)$.



Setning: Anta at $I$ er et intervall og at $f$ er kontinuerlig på $I$ og deriverbar i alle indre punkter i $I$. Hvis $f'(x) \geq 0$ i alle indre punkter i $I$, så er ~~voksende~~ $f$ voksende på $I$. Hvis istedenfor $f'(x) \leq 0$ i alle indre punkter, så er $f$ avtagende på $I$.

Bevis: Anta at $f'(x) \geq 0$ i alle indre punkter og la $x_1, x_2 \in I$, $x_1 < x_2$. Ved MVS:

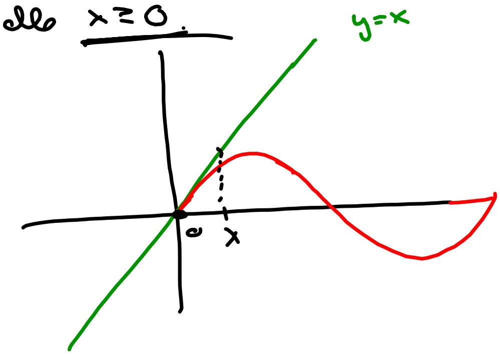
$$\frac{f(x_2)-f(x_1)}{\underbrace{x_2-x_1}_{\vee\!\mid\; 0}} = f'(c) \geq 0,$$




så $f(x_2)-f(x_1) \geq 0$, så $f(x_2) \geq f(x_1)$.

Eksempel: Vis $\sin x \leq x$ for alle $x \geq 0$.

Bruker MVS på intervallet $[0,x]$:




$$\frac{\sin x - \sin 0}{x-0} = \cos c$$

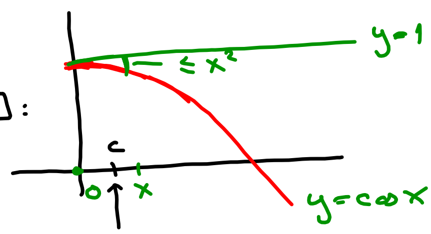
$$\frac{\sin x}{x} = \cos c \quad \Big|\, x$$

$$\sin x = \underbrace{\cos c}\cdot x \leq x.$$

siden $\cos c \leq 1$.

Eksempel: Vis at for $0 \leq x \leq \frac{\pi}{2}$, så er

$$\underbrace{1-\cos x} \leq x^2$$




Bruker MVS på $\cos x$ og intervallet $[0,x]$:

$$\frac{\cos x - \cos 0}{x-0} = -\sin c \qquad (\cos 0 = 1)$$

$\sin c \leq \sin x$

$$\frac{\cos x - 1}{x} = -\sin c \quad \Big|\, x$$

$$\cos x - 1 = -\sin c \cdot x \quad \Big|\, -1$$

$$1-\cos x = \sin c \cdot x \leq \underbrace{\sin x}_{\leq x} x \leq x^2$$

# L'Hôpitals regel

**Ubestemte uttrykk:**

"$\frac{0}{0}$"-uttrykk: $\lim_{x\to a} \frac{f(x)}{g(x)}$ når $f(x)\to 0$, $g(x)\to 0$

"$\frac{\infty}{\infty}$"-uttrykk: $\lim_{x\to a} \frac{f(x)}{g(x)}$ —"— $f(x), g(x) \to \infty$

"$0\cdot\infty$"-uttrykk: $\lim_{x\to a} f(x)g(x)$ $f(x)\to 0$, $g(x)\to\infty$

"$\infty-\infty$"-uttrykk: $\lim_{x\to a} [f(x)-g(x)]$, $f(x), g(x)\to\infty$

"$1^\infty$"-uttrykk: $\lim_{x\to a} f(x)^{g(x)}$, $f(x)\to 1$, $g(x)\to\infty$

"$0^0$"-uttrykk: $\underbrace{\lim f(x)^{g(x)}}_{[e^{\ln f(x)}]^{g(x)} = e^{g(x)\ln f(x)}}$, $f(x)\to 0$, $g(x)\to 0$ $\overset{\uparrow}{0}$ $\infty$

"$\infty^0$"-uttrykk: $f(x)^{g(x)}$ når $f(x)\to\infty$, $g(x)\to 0$.

L'Hôpitals regel tar for seg "$\frac{0}{0}$" og "$\frac{\infty}{\infty}$".

**L'Hôpitals regel:** Anta at enten

(i) $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$

eller

(ii) $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty$

og at $\lim_{x\to a} \frac{f'(x)}{g'(x)} = L$ (der $L$ kan være $\pm\infty$). Da er

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} = L.$$

OK med $a = \pm\infty$

OK $x\to a^+$ eller $x\to a^-$

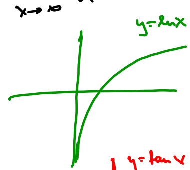
Selv om $\lim \frac{f'(x)}{g'(x)}$ ikke eksisterer, kan det tenkes at $\lim \frac{f(x)}{g(x)}$ likevel finnes.

**Eksempel:** $\lim\limits_{x\to 0} \frac{e^x-1}{x} \overset{L'H}{=} \lim\limits_{x\to 0} \frac{e^x}{1} = \frac{1}{1} = \underline{\underline{1}}$

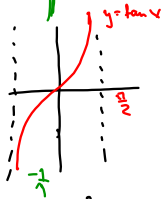
**Eksempel:** $\lim\limits_{x\to 0} \frac{\cos x - 1}{x^2} \overset{L'H}{=} \lim\limits_{x\to 0} \frac{-\sin x}{2x} \overset{L'H}{=} \lim\limits_{x\to 0} \frac{-\cos x}{2} = \underline{\underline{-\frac{1}{2}}}$

**Eksempel** $\lim\limits_{x\to\infty} \frac{e^x}{x^2} \overset{L'H}{=} \lim\limits_{x\to\infty} \frac{e^x}{2x} \overset{L'H}{=} \lim\limits_{x\to\infty} \frac{e^x}{2} = \infty$

**Eksempel:** $\lim\limits_{x\to 0^+} \underset{\downarrow}{\overset{0}{x}}\, \underset{-\infty}{\ln x} = \lim\limits_{x\to 0^+} \frac{\ln x \to -\infty}{\frac{1}{x} \to \infty}$




$\overset{L'H}{=} \lim\limits_{x\to 0^+} \frac{\frac{1}{x}\cdot x^2}{-\frac{1}{x^2}\cdot x^2} = \lim\limits_{x\to 0^+} \frac{x}{-1} = \underline{\underline{0}}$




**Eksempel:** $\lim\limits_{x\to \frac{\pi}{2}^-} (x-\frac{\pi}{2})\tan x \quad (0\cdot\infty)$

$= \lim\limits_{x\to\frac{\pi}{2}^-} \frac{\tan x}{\frac{1}{x-\frac{\pi}{2}}} \overset{L'H}{=} \lim\limits_{x\to\frac{\pi}{2}^-} \frac{\frac{1}{\cos^2 x}}{-\frac{1}{(x-\frac{\pi}{2})^2}}$

$= \lim\limits_{x\to\frac{\pi}{2}^-} -\frac{1}{\cos^2 x}\cdot\frac{(x-\frac{\pi}{2})^2}{1} = -\lim\limits_{x\to\frac{\pi}{2}^-} \frac{(x-\frac{\pi}{2})^2}{\cos^2 x} = -\lim\limits_{x\to\frac{\pi}{2}^-} \left(\frac{x-\frac{\pi}{2}}{\cos x}\right)^2 = -(-1)^2 = \underline{\underline{-1}}$

**Mellomregning:** $\lim\limits_{x\to\frac{\pi}{2}^-} \frac{x-\frac{\pi}{2}}{\cos x} \overset{L'H}{=} \lim\limits_{x\to\frac{\pi}{2}^-} \frac{1}{-\sin x} = \underline{\underline{-1}}$

**Eksempel:** $\lim\limits_{x\to\infty} (x^2\cos\frac{1}{x} - x^2) = \lim\limits_{x\to\infty} x^2(\cos\frac{1}{x}-1) \quad (\infty - \infty,\ \infty\cdot 0)$

$= \lim\limits_{x\to\infty} \frac{\cos\frac{1}{x}-1}{\frac{1}{x^2}} \overset{L'H}{=} \lim\limits_{x\to\infty} \frac{+\sin(\frac{1}{x})(+\frac{1}{x^2})\cdot x^3}{-2x^{-3}\cdot x^3} = \lim\limits_{x\to\infty} \frac{x\cdot\sin\frac{1}{x}}{-2}$

$= -\frac{1}{2}\lim\limits_{x\to\infty} \frac{\sin\frac{1}{x}}{\frac{1}{x}} \overset{L'H}{=} -\frac{1}{2}\lim\limits_{x\to\infty} \frac{\cos\frac{1}{x}\,(-\cancel{\frac{1}{x^2}})}{-\cancel{\frac{1}{x^2}}} = \underline{\underline{-\frac{1}{2}}}$

**Eksempel:** $\lim\limits_{x\to\infty} \left(\underbrace{1+\frac{2}{x}}_{1}\right)^{x \to\infty} = \lim\limits_{x\to\infty} \left(e^{\ln(1+\frac{2}{x})}\right)^x$

$= \lim\limits_{x\to\infty} e^{\overset{\infty}{x}\,\overset{0}{\ln(1+\frac{2}{x})}} = e^2$

**Mellomregning:** $\lim\limits_{x\to\infty} x\ln(1+\frac{2}{x}) = \lim\limits_{x\to\infty} \frac{\ln(1+\frac{2}{x})}{\frac{1}{x}}$

$\overset{L'H}{=} \lim\limits_{x\to\infty} \frac{\frac{1}{1+2/x}\,(+\cancel{\frac{2}{x^2}})}{+\cancel{\frac{1}{x^2}}} = \lim\limits_{x\to\infty} \frac{\frac{2}{1+\frac{2}{x}}}{1} = \underline{\underline{2}}$

**Eksempel:** $\lim\limits_{x\to 0^+} (\sin 2x)^x \quad 0^0$

$= \lim\limits_{x\to 0^+} \left(e^{\ln(\sin 2x)}\right)^x = \lim\limits_{x\to 0^+} e^{x\ln(\sin 2x)} = e^0 = \underline{1}$

**Mellomregning:** $\lim\limits_{x\to 0^+} x\ln(\sin 2x) = \lim\limits_{x\to 0^+} \frac{\ln(\sin 2x)}{\frac{1}{x}}$

$\overset{L'H}{=} \lim\limits_{x\to 0^+} \frac{\frac{1}{\sin 2x}\,\cos 2x\cdot 2}{-\frac{1}{x^2}} = -\underbrace{\lim\limits_{x\to 0^+} \cos 2x}_{=1}\ \lim\limits_{x\to 0^+} \frac{2}{\sin 2x}\,x^2$

$= -1\lim\limits_{x\to 0^+} \underbrace{\frac{2x}{\sin 2x}}_{1}\cdot \underbrace{x}_{0} = \underline{\underline{0}}$

$\frac{\sin u}{u} \to 1$

$\frac{u}{\sin u} \to 1$