Delvis integrasjon

$$\int_a^b u v' \, dx = \Big[uv\Big]_a^b - \int_a^b u' v \, dx$$

Eksempel: $\int_0^{\pi} x \cos x \, dx$

$u = x$, $v' = \cos x$
$u' = 1$, $v = \sin x$

$$= \Big[x \sin x\Big]_0^{\pi} - \int_0^{\pi} 1 \cdot \sin x \, dx$$

$$= \pi \underbrace{\sin \pi}_{=0} - 0 \cdot \sin 0 + \Big[\cos x\Big]_0^{\pi}$$

$$= 0 + \Big[\cos \pi - \cos 0\Big] = -1 - 1 = -2$$

Rekursjonsformler:

Eksempel: $I_n = \int x^n e^{-x} \, dx$

$u = x^n$, $v' = e^{-x}$
$u' = n x^{n-1}$, $v = -e^{-x}$

$$= -x^n e^{-x} - \int n x^{n-1} (-e^{-x}) \, dx$$

$$= -x^n e^{-x} + n \underbrace{\int x^{n-1} e^{-x} \, dx}_{I_{n-1}}$$

Altså:

$$I_n = -x^n e^{-x} + n I_{n-1} \qquad (*)$$

Anvendelse: Finn $I_3 = \int x^3 e^{-x} \, dx$

Bruker $(*)$ for $n = 3$:

$$I_3 = -x^3 e^{-x} + 3 \cdot I_2 \qquad \text{(Bruker (*) med } n=2\text{)}$$

$$= -x^3 e^{-x} + 3(-x^2 e^{-x} + 2 I_1)$$

$$= -x^3 e^{-x} - 3x^2 e^{-x} + 6 I_1 \qquad \text{(Bruker (*) med } n=1\text{):}$$

$$= -x^3 e^{-x} - 3x^2 e^{-x} + 6(-x e^{-x} + 1 \cdot I_0) \qquad /\!/ \int e^{-x} \, dx$$

$$= -x^3 e^{-x} - 3x^2 e^{-x} - 6x e^{-x} + 6 \int \underbrace{x^0}_{1} e^{-x} \, dx$$

$$= -x^3 e^{-x} - 3x^2 e^{-x} - 6x e^{-x} - 6 e^{-x} + C$$

Eksempel: $I_n = \int_0^{\pi} \sin^n x \, dx = \int_0^{\pi} \sin^{n-2} x \sin^2 x \, dx$

$$= \int_0^{\pi} \sin^{n-2} x (1 - \cos^2 x) \, dx = \underbrace{\int_0^{\pi} \sin^{n-2} x \, dx}_{I_{n-2}} - \int_0^{\pi} \sin^{n-2} x \cos^2 x \, dx$$

$$= I_{n-2} - \int_0^{\pi} \sin^{n-2} x \cos^2 x \, dx$$

$u = \cos x$, $v' = \sin^{n-2} x \cos x$
$u' = -\sin x$, $v = \frac{1}{n-1} \sin^{n-1} x$

$$= I_{n-2} - \int_0^{\pi} \underbrace{\sin^{n-2} x \cos x}_{v'} \cdot \underbrace{\cos x}_{u} \, dx$$

$$= I_{n-2} - \left(\Big[\frac{1}{n-1} \underbrace{\sin^{n-1} x}_{=0} \cos x\Big]_0^{\pi} - \int_0^{\pi} (-\sin x) \frac{1}{n-1} \sin^{n-1} x \, dx \right)$$

$$= I_{n-2} - \frac{1}{n-1} \underbrace{\int_0^{\pi} \sin^n x \, dx}_{I_n} = I_{n-2} - \frac{1}{n-1} I_n$$

Altså $I_n = I_{n-2} - \frac{1}{n-1} I_n$

$$\underbrace{\left(1 + \frac{1}{n-1}\right)}_{} I_n = I_{n-2}$$

$$\frac{n}{n-1} = \frac{n - \cancel{1} + \cancel{1}}{n-1} \qquad I_n = \frac{n-1}{n} I_{n-2}$$

Eksempel: $I_4 = \int_0^{\pi} \sin^4 x \, dx$

Substitusjon (seksjon 9.2)

$u = g(x)$

$du = \underline{g'(x)\,dx}$

Enkel form: $\int f(g(x))\, g'(x)\, dx = \int f(u)\, du$

↑ magisk faktor

Ekle form: $\int f(g(x))\, dx$

Magisk triks: La h være den omvendte funksjonen til g:

$$h'(g(x)) = \frac{1}{g'(x)} \quad \Big|\, g'(x)$$

$h(y) = \frac{1}{g'(x)}$

$\Downarrow \quad y = g(x)$

$$\underline{h'(g(x))\, g'(x) = 1}$$

magisk faktor

Ekle form. $\int f(g(x)) \cdot 1\, dx = \int f(g(x))\, \underline{h'(g(x))}\, \underbrace{g'(x)\, dx}$

$u = g(x)$

$du = g'(x)\, dx$

$$= \int f(u)\, \underbrace{h'(u)}\, du.$$

↑ pris vi må betale for å bli $g(x)$.

Praksis: $\int f(g(x))\, dx = \int f(u)\, h'(u)\, du$

$u = g(x)$ Løser for x.

$\Downarrow$

$\underline{x = h(u)}$

$\frac{dx}{du} = h'(u)$

$dx = \underline{h'(u)\, du}$

Eksempel: $\int \frac{1}{1+\sqrt{x}}\, dx$

$\underline{u = \sqrt{x}}$

$x = u^2$

$\frac{dx}{du} = 2u$

$dx = 2u\, du$

$$= \int \frac{1}{1+u}\, \underline{2u}\, du$$

pris

$$= 2\int \frac{u}{1+u}\, du = 2\int \frac{u+1-1}{u+1}\, du$$

$$= 2\int \left(1 - \frac{1}{u+1}\right) du = 2\left(u - \ln|u+1|\right) + C$$

$$= 2\left(\sqrt{x} - \ln|\sqrt{x}+1|\right) + C$$

Eksempel: $\int \cos\sqrt{x}\, dx = \int \cos u\; 2u\, du$

$\underline{u = \sqrt{x}}$

$x = u^2$

$dx = 2u\, du$

$$= 2\int u\cos u\, du = 2\left[u\sin u - \int \sin u\right]$$

$U = u \qquad V' = \cos u$

$\underline{U' = 1} \qquad \underline{V = \sin u}$

$$= 2\left[u\sin u + \cos u\right] + C$$

$$= 2\left[\sqrt{x}\sin\sqrt{x} + \cos\sqrt{x}\right] + C$$

Eksempel: $I = \int e^{\arcsin x}\,dx$

$= \int e^u \cos u \, du$

$u = \arcsin x$
$x = \sin u$
$\frac{dx}{du} = \cos u$
$dx = \cos u \, du$

Mellomregning:

$I_1 = \int e^u \cos u \, du = e^u \cos u + \int e^u \sin u \, du$

$U = \cos u \quad V' = e^u$
$U' = -\sin u, \quad V = e^u$

$= e^u \cos u + e^u \sin u - \int e^u \cos u \, du$

$U = \sin u \quad V' = e^u$
$U' = \cos u, \quad V = e^u$

$= e^u \cos u + e^u \sin u - I_1$

Altså $I_1 = e^u \cos u + e^u \sin u - I_1$

$2I_1 = e^u \cos u + e^u \sin u + C$

$I_1 = \frac{e^u}{2}(\cos u + \sin u) + C$

Tilbake til I:

$u = \arcsin x, \quad x = \sin u$

$I = \int e^{\arcsin x}\,dx = \underbrace{\int e^u \cos u \, du}_{I_1}$

$= \frac{e^u}{2}(\cos u + \sin u) + C = \frac{e^{\arcsin x}}{2}\left(\sqrt{1-x^2} + x\right) + C$

$\cos u = \pm\sqrt{1-\sin^2 u}$
$= \sqrt{1-x^2}$

Substitusjon i bestemte integraler:

$$\int_a^b f(g(x))\,dx = \int_{g(a)}^{g(b)} f(u)\,h'(u)\,du$$

$u = g(x)$
$x = h(u)$
$dx = h'(u)\,du$

Eksempel: $\int_1^3 \arctan \sqrt{x}\,dx$

$u = \sqrt{x}$
$x = u^2$
$dx = 2u\,du$
Når $x=1$, er $u = \sqrt{1} = 1$
Når $x=3$, er $u = \sqrt{3}$

$= \int_1^{\sqrt{3}} \arctan u \cdot 2u \, du$

$U = \arctan u \quad V' = 2u$
$U' = \frac{1}{1+u^2} \quad V = u^2$

$= \left[u^2 \arctan u\right]_1^{\sqrt{3}} - \int_1^{\sqrt{3}} \frac{u^2}{1+u^2}\,du$

$= [3 \cdot \underbrace{\arctan\sqrt{3}}_{=\frac{\pi}{3}} - 1 \cdot \underbrace{\arctan 1}_{=\frac{\pi}{4}}] - \int_1^{\sqrt{3}} \frac{u^2+1-1}{u^2+1}\,du$

$= [\pi - \frac{\pi}{4}] - \int_1^{\sqrt{3}} \left(1 - \frac{1}{u^2+1}\right) du = \frac{3\pi}{4} - \left[u - \arctan u\right]_1^{\sqrt{3}}$

$= \frac{3\pi}{4} - [(\sqrt{3} - \underbrace{\arctan\sqrt{3}}_{=\frac{\pi}{3}}) - (1 - \underbrace{\arctan 1}_{=\frac{\pi}{4}})]$

$= \frac{3\pi}{4} - \sqrt{3} + \frac{\pi}{3} + 1 - \frac{\pi}{4} = \frac{\pi}{2} + \frac{\pi}{3} + 1 - \sqrt{3} = \frac{5\pi}{6} + 1 - \sqrt{3}$

Eksempel: $\int_0^3 \sqrt{9-x^2}\,dx$

$x = 3\sin u$

$dx = 3\cos u\,du$

Idé $\sqrt{1-\sin^2 u} = \cos u$

$\frac{dx}{du} = 3\cos u$

$$\cancel{= \int_0^{\frac{\pi}{2}} \sqrt{9-9\sin^2 u}\,du = \int_0^{\pi/2} 3\underbrace{\sqrt{1-\sin^2 u}}_{\cos^2 u}\,du = 3\int_0^{\pi/2} \cos u\,du}$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{9-9\sin^2 u}\;3\cos u\,du = \int_0^{\pi/2} 3\underbrace{\sqrt{1-\sin^2 u}}_{\cos u}\;3\cos u\,du$$

$$:= 9\int_0^{\pi/2} \cos^2 u\,du = \cancel{A}$$

$$= 9\Big(\underbrace{[\cos u \sin u]_0^{\pi/2}}_{=0} + \int_0^{\pi/2} \sin^2 u\,du\Big)$$

$$= 9\Big(\int_0^{\pi/2}(1-\cos^2 u)\,du\Big)$$

$$= 9\Big(\frac{\pi}{2} - \int_0^{\pi/2}\cos^2 u\,dx\Big)$$

$$\frac{9\pi}{4}$$

Delvis integrasjon

$u = \cos u \qquad v' = \cos u$

$u' = -\sin u \qquad v = \sin u$

Dette betyr at

$$9\int_0^{\pi/2}\cos^2 u\,du = \frac{9\pi}{2} - 9\int_0^{\pi/2}\cos^2 u\,du$$

$$2\cdot 9\int_0^{\pi/2}\cos^2 u\,du = \frac{9\pi}{2}$$

$$9\int_0^{\pi/2}\cos^2 u\,du = \frac{9\pi}{4}$$

Delbrøk oppspalting

Kan fra før: $\int \frac{1}{x^2+x-6}\,dx$

$x^2+x-6 = (x+3)(x-2)$

Hva hvis denne ikke kan faktoriseres.

$$\frac{1}{x^2+x-6} = \frac{1}{(x+3)(x-2)} = \frac{A}{x+3} + \frac{B}{x-2}$$

$$\int \frac{1}{x^2+1}\,dx = \arctan x + C$$