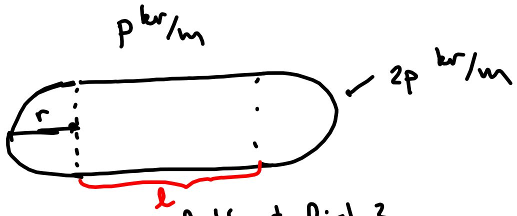
Uoppstilte oppgaver

Eksamen 2007. Speil: Areal $1\,m^2$



Hvilken radius r gir billigst list?

Pris på list:

$$K = \underbrace{2pl}_{\text{rett list}} + \underbrace{2p \cdot 2\pi r}_{\text{krum list}}$$

Areal: $1 = \underbrace{2rl}_{\text{rektangel}} + \underbrace{\pi r^2}_{\text{halvsirkler}} \xrightarrow{\text{løser for } l} 2rl = 1 - \pi r^2$

$$\Rightarrow l = \frac{1-\pi r^2}{2r} = \underline{\frac{1}{2r} - \frac{\pi}{2} r}$$

Kostnad: $K(r) = \cancel{2}p\left(\frac{1}{\cancel{2}r} - \frac{\pi}{\cancel{2}}r\right) + 4p\pi r$

$$= \frac{p}{r} - p\pi r + 4p\pi r = \underline{p\left(\frac{1}{r} + 3\pi r\right)}$$

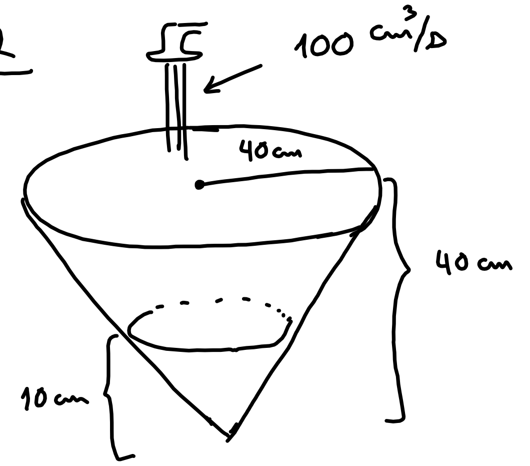
Deriverer mhp r:

$$K'(r) = p\left(-\frac{1}{r^2} + 3\pi\right)$$

Løser. $0 = K'(r) = p\left(\underbrace{-\frac{1}{r^2} + 3\pi}\right) \Rightarrow \frac{1}{r^2} = 3\pi \Rightarrow r^2 = \frac{1}{3\pi}$

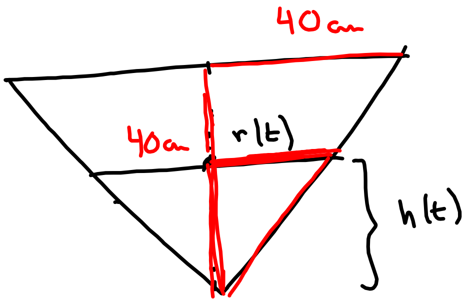
$$\Rightarrow \underline{\underline{r = \frac{1}{\sqrt{3\pi}}}}$$

Oppgave 7.2.2



Hvor mye øker høyden med når $h = 10$ cm?

Igjennom $V'(t)$, søker $h'(t)$.



$$V(t) = \frac{1}{3} gh = \frac{1}{3}\pi r(t)^2 h(t)$$

$$\frac{r(t)}{h(t)} = \frac{40}{40} \Rightarrow r(t) = h(t)$$

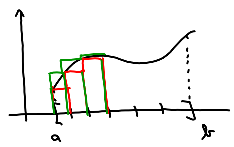
$$V(t) = \frac{1}{3}\pi h(t)^3$$

Deriverer: $V' = \frac{\pi}{\cancel{3}} \cancel{3} h^2 \cdot h' = \pi h^2 h'$

Løser for h': $h' = \frac{V'}{\pi h^2}$

Når $h = 10$: $h' = \frac{\cancel{100}}{\pi \cdot \cancel{10^2}} = \frac{1}{\pi}$ cm/s

Integrasjon



Øvre trappesum: $\Phi(\Pi)$ Øvreintegral: $\overline{\int_a^b} f(x)\,dx = \inf\{\Phi(\Pi) : \Pi \text{ er part}\}$

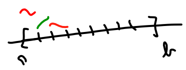
Nedre trappesum: $N(\Pi)$ Nedreintegral: $\underline{\int_a^b} f(x)\,dx = \sup\{N(\Pi) : \text{—"—}\}$

f er <u>integrerbar</u> dersom $\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx$ og i så fall er

$$\int_a^b f(x)\,dx = \overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx$$

$$f(x) = \begin{cases} g(x) & x_0 < x < x_1 \\ h(x) & x_1 < x < x_2 \end{cases}$$

Når er f integrerbar?



(i) f monoton

(ii) f kontinuerlig

(iii) Hvis $a = x_0 < x_1 < x_2 < \ldots < x_n = b$

Dersom f er integrerbar på hver $[x_{i-1}, x_i]$, så er den integrerbar.

Analysens fundamentalteorem: Hvis $f : [a,b] \to \mathbb{R}$ er kontinuerlig, så er f integrerbar og

$$F(x) = \int_a^x f(t)\,dt$$

er en antiderivert til f. Hvis K er en (vilkårlig) antiderivert til f, så er

$$\int_a^b f(t)\,dt = K(b) - K(a)$$

Eksempel: $G(x) = \int_0^x \frac{\sin t}{t^2+1}\,dt$. Finn $G'(x)$.

Ved AF er $G(x)$ en antiderivert til $\frac{\sin x}{x^2+1}$, så

$$G'(x) = \frac{\sin x}{x^2+1}$$

Hva hvis

$$F(x) = \int_{g(x)}^{h(x)} f(t)\,dt,$$

hva blir nå $F'(x)$? La $K(x)$ være en antiderivert til $f(x)$. Da er

$$\int_a^b f(t)\,dt = K(b) - K(a) \quad \text{for alle } a, b.$$

Setter $a = g(x)$, $b = h(x)$ og får

$$F(x) = \int_{g(x)}^{h(x)} f(t)\,dt = K(h(x)) - K(g(x))$$

Deriverer:

$$F'(x) = K'(h(x))h'(x) - K'(g(x))g'(x)$$
$$= f(h(x))h'(x) - f(g(x))g'(x)$$

dvs

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = \underline{\underline{f(h(x))h'(x) - f(g(x))g'(x)}}$$

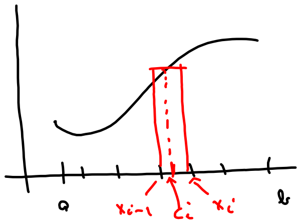
Eksempel (kont 2014):

$$\lim_{x\to 0^+} \frac{\int_0^{x^2} \sqrt{t}\,e^t\,dt}{x^3} = \text{"}\frac{0}{0}\text{"}$$

$$\overset{\text{L'H}}{=} \lim_{x\to 0^+} \frac{\left(\sqrt{x^2}\,e^{x^2}\right)2x}{3x^2} = \lim_{x\to 0^+} \frac{\cancel{x}\,e^{x^2}\cdot 2\cancel{x}}{3\cancel{x^2}} = \frac{2}{3}\lim_{x\to 0^+} e^{x^2}$$
$$= \underline{\underline{\frac{2}{3}}}$$

Riemannsummer:

$$\sum f(c_i)(x_i - x_{i-1}) \to \int_a^b f(x)\,dx$$



Anvendelser:

(i) Omdreiningslegeme om x-aksen:

$$V = \pi \int_a^b f(x)^2\,dx$$

(ii) —— " —— om y-aksen

$$V = 2\pi \int_a^b x f(x)\,dx$$

(iii) Buelengde

$$L = \int_a^b \sqrt{1 + f'(x)^2}\,dx$$

Integrasjonsteknikker:

(i) Delvis integrasjon: $\int u v'\,dx = uv - \int u' v\,dx$

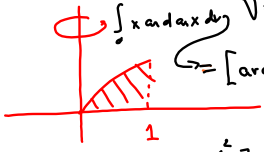
(ii) Substitusjon: enkel $\int f(g(x))\,g'(x)\,dx = \int f(u)\,du$ $\quad u = g(x)$, $du = g'(x)\,dx$

vanskelig $\int f(g(x))\,dx = \int f(u)\,h'(u)\,du$. $\quad u = g(x)$, $x = h(u)$, $dx = h'(u)\,du$

(iii) Delbrøksoppspalting: $\int \frac{P(x)}{Q(x)}\,dx$ $\quad P, Q$ − polynomer.

Oppgave: Finn volumet som fremkommer når arealet under grafen til $f(x) = \arctan x$, $0 \le x \le 1$ dreies om y-aksen:

$$V = 2\pi \int_a^b x f(x)\,dx = 2\pi \int_0^1 x \arctan x\,dx$$



$$\int_0^1 x \arctan x\,dx = \left[\arctan x \cdot \frac{x^2}{2}\right]_0^1 - \int_0^1 \frac{\frac{x^2}{2}}{1+x^2}\,dx$$

$u = \arctan x \quad v' = x$

$u' = \frac{1}{1+x^2}, \quad v = \frac{x^2}{2}$

$$= \left[\underbrace{\arctan 1}_{\pi/4} \cdot \frac{1^2}{2}\right] - \frac{1}{2}\int_0^1 \frac{x^2}{x^2+1}\,dx$$

$$= \frac{\pi}{8} - \frac{1}{2}\int_0^1 \frac{(x^2+1)-1}{x^2+1}\,dx = \frac{\pi}{8} - \frac{1}{2}\int_0^1 \left(1 - \frac{1}{x^2+1}\right)dx$$

$$= \frac{\pi}{8} - \frac{1}{2}\left[x - \arctan x\right]_0^1 = \frac{\pi}{8} - \frac{1}{2}\left[1 - \frac{\pi}{4}\right]$$

$$= \frac{\pi}{8} - \frac{1}{2} + \frac{\pi}{8} = \frac{\pi}{4} - \frac{1}{2}$$

$$V = 2\pi I = \frac{\pi^2}{2} - \pi$$

Eksempel: $I = \int_4^9 e^{\sqrt{t}}\,dt$

$u = \sqrt{t} \Rightarrow u^2 = t$

$dt = 2u\,du$

$t = 4: \ u = \sqrt{4} = 2$

$t = 9: \ u = \sqrt{9} = 3$

$$= \int_2^3 e^u\,2u\,du$$

$U = 2u \quad V' = e^u$

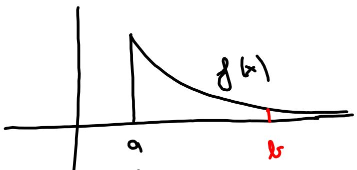
$U' = 2, \quad V = e^u$

$$= \left[2u e^u\right]_2^3 - \int_2^3 2e^u\,du$$

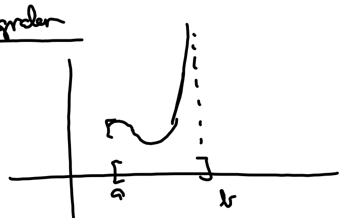
$$= 6e^3 - 4e^2 - 2\left[e^u\right]_2^3$$

$$= 6e^3 - 4e^2 - 2e^3 + 2e^2 = 4e^3 - 2e^2$$

Uegentlige integraler



$$\int_a^\infty f(x)\,dx$$



$$\int_a^b f(x)\,dx$$

Definisjon:

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx = \begin{cases}\text{eksisterer (som et tall): konvergerer}\\ \text{ikke eksisterer: divergerer}\end{cases}$$

Eksempel: $\int_0^\infty \frac{1}{1+x^2}\,dx = \lim_{b\to\infty}\int_0^b \frac{1}{1+x^2}\,dx = \lim_{b\to\infty}\left[\arctan x\right]_0^b$

$= \lim_{b\to\infty} \arctan b = \frac{\pi}{2}$ (konvergens)

Teorem: $\int_1^\infty \frac{1}{x^p}\,dx = \begin{cases}\text{konvergerer for } p>1\\ \text{divergerer for } p\leq 1.\end{cases}$

Grensesammenligningskriteriet: Anta at $f(x), g(x) \geq 0$ og at $\int_a^\infty f(x)\,dx$ konvergerer. Hvis

$$\lim_{x\to\infty}\frac{g(x)}{f(x)} < \infty,$$

så konvergerer også $\int_a^\infty g(x)\,dx$.

Hvis $\int_a^\infty f(x)\,dx$ divergerer og

$$\lim_{x\to\infty}\frac{g(x)}{f(x)} > 0,$$

så divergerer også $\int_a^\infty g(x)\,dx$

Eksempel: Avgjør om $\int_1^\infty \frac{x^3+2x+7}{x^5+2x^2+3}\,dx$ divergerer eller konvergerer. — $g(x)$

$$\frac{x^3+2x+7}{x^5+2x^2+3} = \frac{\cancel{x^3}\left(1+\frac{2}{x^2}+\frac{7}{x^3}\right)}{x^2\cancel{x^3}\left(1+\frac{2}{x^3}+\frac{3}{x^5}\right)} \sim \frac{1}{x^2} \text{ — } f(x)$$

Vet at $\int_1^\infty \frac{1}{x^2}\,dx$ konvergerer. Regner ut

$$\lim_{x\to\infty}\frac{g(x)}{f(x)} = \lim_{x\to\infty}\frac{\frac{x^3+2x+7}{x^5+2x^3+3}}{\frac{1}{x^2}} = \lim_{x\to\infty}\frac{\cancel{\frac{1}{x^2}}\,\frac{\left(1+\frac{2}{x^2}+\frac{7}{x^3}\right)}{\left(1+\frac{2}{x^3}+\frac{3}{x^5}\right)}}{\cancel{\frac{1}{x^2}}} = 1 < \infty$$

Altså konvergerer $\int g(x)\,dx$.