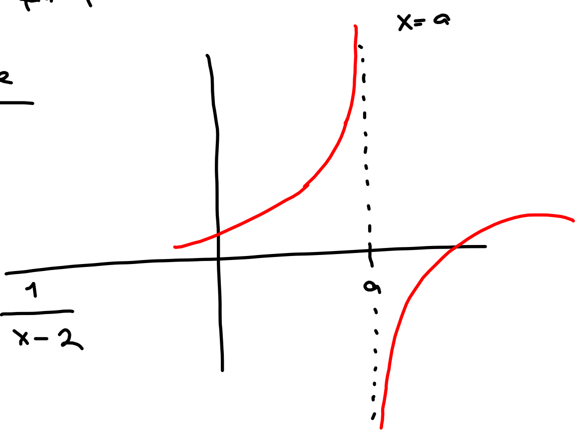
Asymptoter (6.5)

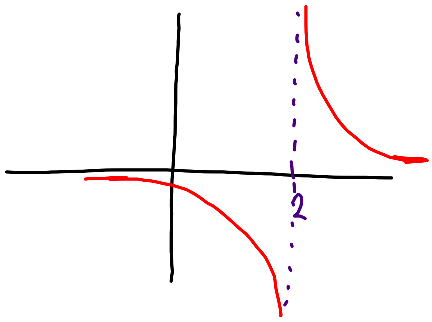
Hva skjer når grafen forsvinner ut av arket?

Vertikal asymptote

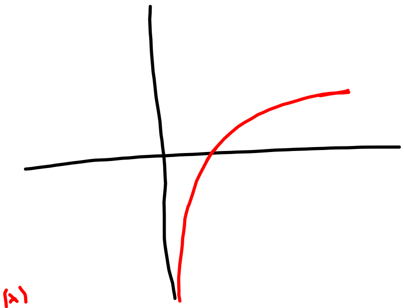
$$\lim_{x\to a} f(x) = \pm \infty$$



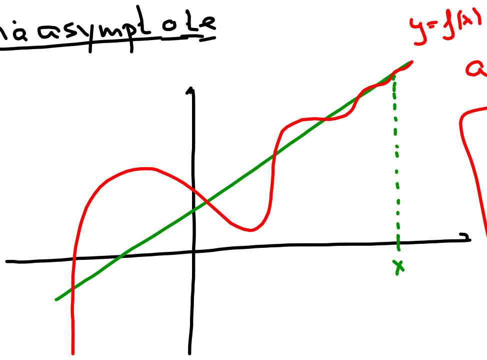
Eksempel: $f(x) = \frac{1}{x-2}$

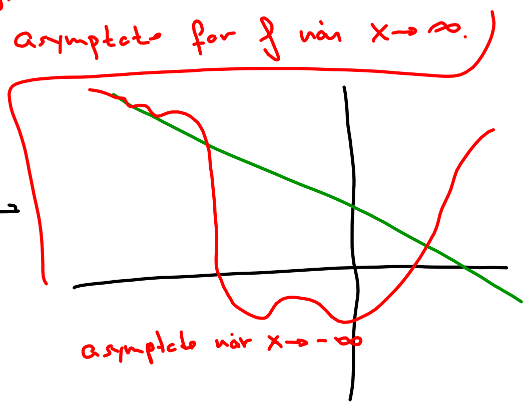


Eksempel: $f(x) = \ln x$



Skråasymptote

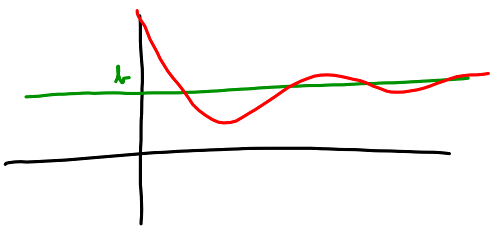




Definisjon: Linjen $y = ax + b$ er en asymptote for $f(x)$ når x går mot ∞ dersom

$$\lim_{x\to\infty} [f(x) - (ax+b)] = 0$$

Spesialtilfelle: Horisontal asymptote



$y = b = 0 \cdot x + b$

Hvordan finner vi asymptoter når de finnes?

Anta at f har en asymptote $y = ax + b$, men at vi ikke vet hva a og b er. Hvordan kan vi finne dem?

Vet at $\lim_{x\to\infty} [\underbrace{f(x) - ax - b}] = 0$

Dele på x

$$0 = \lim_{x\to\infty} \frac{f(x) - ax - b}{x} = \lim_{x\to\infty} \left[\frac{f(x)}{x} - a - \frac{b}{x}\right]$$

$$= \lim_{x\to\infty} \frac{f(x)}{x} - a \Rightarrow \lim_{x\to\infty} \frac{f(x)}{x} = a$$

Siden vi skal ha $\lim_{x\to\infty} [f(x) - ax - b] = 0$, må vi ha

$\lim_{x\to\infty} [f(x) - ax] = b$.

Algoritme for å finne asymptote:

(i) Regn ut $\lim_{x\to\infty} \frac{f(x)}{x}$. Hvis denne grenseverdien ikke eksisterer, så har vi ingen asymptote. Hvis $\lim_{x\to\infty} \frac{f(x)}{x} = a$, så

(ii) Regn ut $\lim_{x\to\infty} [f(x) - ax]$. Hvis denne grenseverdien ikke eksisterer, så er det ingen asymptote. Hvis $\lim_{x\to\infty} [f(x) - ax] = b$, så er $y = ax + b$ en asymptote.

Eksempel (ekkelt): Undersøk om $f(x) = \sqrt[3]{8x^3 + x^2}$ har en asymptote når $x \to \infty$.

(i) $$\lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{(8x^3 + x^2)^{1/3}}{x} = \lim_{x\to\infty} \frac{\cancel{x}\left(8 + \frac{1}{x}\right)^{1/3}}{\cancel{x}}$$

$$= \lim_{x\to\infty} \left(8 + \frac{1}{x}\right)^{1/3} = 8^{1/3} = 2$$

$\boxed{a = 2.}$

(ii) $$\lim_{x\to\infty} [f(x) - ax] = \lim_{x\to\infty} [\underbrace{(8x^3 + x^2)^{1/3}}_{\infty} - \underbrace{2x}_{\infty}]$$

$$= \lim_{x\to\infty} \left[x\left(8 + \frac{1}{x}\right)^{1/3} - 2x\right] = \lim_{x\to\infty} \underbrace{x}_{\to\infty} \underbrace{\left(\left(8 + \frac{1}{x}\right)^{1/3} - 2\right)}_{\to 0}$$

$$= \lim_{x\to\infty} \frac{\overbrace{\left(8 + \frac{1}{x}\right)^{1/3} - 2}^{\to 0}}{\underbrace{\frac{1}{x}}_{\to 0}} \overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{1}{3}\left(8 + \frac{1}{x}\right)^{1/3 - 1} \cancel{\left(-\frac{1}{x^2}\right)}}{\cancel{\left(-\frac{1}{x^2}\right)}}$$

$$= \lim_{x\to\infty} \frac{1}{3}\left(8 + \frac{1}{x}\right)^{-2/3} = \frac{1}{3} 8^{-2/3} = \frac{1}{3} \frac{1}{8^{2/3}}$$

$$= \frac{1}{3} \frac{1}{(\underbrace{8^{1/3}}_{2})^2} = \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}$$

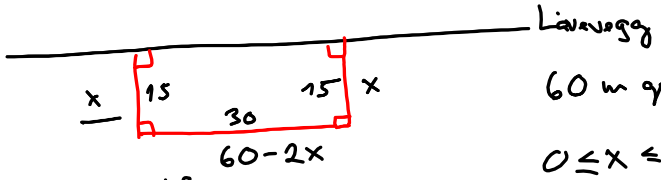
$\boxed{b = \frac{1}{12}}$

Sammenfatning: $y = 2x + \frac{1}{12}$ er asymptote til $f(x) = \sqrt[3]{8x^3 + x^2}$ når $x \to \infty$.

Uoppstilte maks/min-problemer

optimeringsproblemer

Eksempel:



60 m gjerde

$0 \le x \le 30$

Hvordan få størst areal?

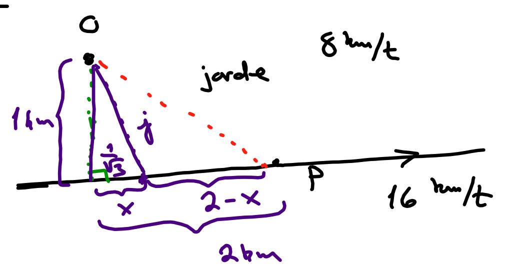
$$A(x) = x(60-2x) = 60x - 2x^2$$

Deriverer: $A'(x) = 60 - 4x$

$$0 = A'(x) \Rightarrow 60 - 4x \Rightarrow 4x = 60 \Rightarrow \underline{\underline{x = 15}}$$

Eksempel: Orienteringsløper

Kortest mulig tid.



$v = \frac{s}{t}$

$t = \frac{s}{v}$

$$j(x) = \sqrt{1^2 + x^2} = \sqrt{1+x^2} \; : \; t_1(x) = \frac{\sqrt{1+x^2}}{8}$$

$$s(x) = 2 - x \; : \; t_2(x) = \frac{2-x}{16}$$

Total tid:

$$t(x) = t_1(x) + t_2(x) = \frac{\sqrt{1+x^2}}{8} + \frac{2-x}{16}$$

Deriverer: $t'(x) = \frac{1}{8} \cdot \frac{1}{\cancel{2}\sqrt{1+x^2}} \cdot \cancel{2}x + \frac{1}{16}(-1)$

$$= \frac{x}{8\sqrt{1+x^2}} - \frac{1}{16}$$

Setter $t'(x) = 0$:

$$\frac{x}{8\sqrt{1+x^2}} - \frac{1}{16} = 0 \quad | \cdot 16$$

$$\frac{2x}{\sqrt{1+x^2}} = 1 \quad | \cdot \sqrt{1+x^2}$$

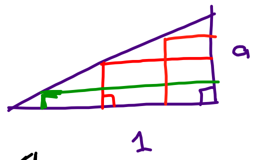
$$2x = \sqrt{1+x^2}$$

Kvadrer:

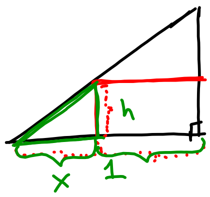
$$4x^2 = 1 + x^2 \Rightarrow 3x^2 = 1 \Rightarrow x^2 = \frac{1}{3} \Rightarrow x = \pm \frac{1}{\sqrt{3}}$$

Vår løsning: $x = \underline{\underline{\frac{1}{\sqrt{3}}}} = \frac{\sqrt{3}}{3}$

Eksempel



Hva er det største arealet rektanglet kan ha?



$A(x) = \text{Areal} = (1-x)h = \underline{ax(1-x) = a(x-x^2)}$

Finner h: $\frac{h}{x} = \frac{a}{1} \Rightarrow h = ax$

Deriverer

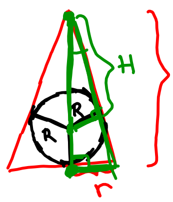
$A'(x) = a(1-2x)$, er 0 når $x = \frac{1}{2}$.

Maks areal $A(\frac{1}{2}) = a \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{a}{4}$

Areal av trekant: $\frac{1}{2} \cdot 1 \cdot a = \frac{a}{2}$

Eksempel:

Kule med radius R. Innskriver den i en kjegle. Hva er det minste volumet kjeglen kan ha.



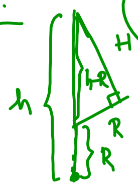
$V = \frac{1}{3} gh = \frac{1}{3}\pi r^2 h$

Sammenheng mellom r og h:

Formlike trekanter:

$\frac{H}{R} = \frac{h}{r}$

Hva er H?



Pytagoras: $(h-R)^2 = H^2 + R^2$

$h^2 - 2hR + \cancel{R^2} = H^2 + \cancel{R^2}$

$H = \sqrt{h^2 - 2hR}$

$\frac{\sqrt{h^2-2hR}}{R} = \frac{h}{r}$ kvadrerer

$\frac{h^2 - 2hR}{R^2} = \frac{h^2}{r^2} \quad | \, r^2 R^2$

$r^2(h^2 - 2hR) = h^2 R^2$

$r^2 = \frac{h^2 R^2}{h^2 - 2hR} = \frac{hR^2}{h-2R}$

Går tilbake til volumformelen:

$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \frac{hR^2}{h-2R} h = \frac{\pi R^2}{3} \frac{h^2}{h-2R}$

Deriverer V mhp h:

$V'(h) = \frac{\pi R^2}{3} \frac{2h(h-2R) - h^2 \cdot 1}{(h-2R)^2} = \frac{\pi R^2}{3} \frac{2h^2 - 4hR - h^2}{(h-2R)^2}$

$= \frac{\pi R^2}{3} \frac{h^2 - 4hR}{(h-2R)^2}$

Løser $V'(h) = 0$: $h^2 - 4hR = 0 \Rightarrow \underline{h = 4R}$

Kjeglen skal være dobbelt så høy som kula