Delbrøkoppspalting

Integrasjon av rasjonale funksjoner: $R(x) = \frac{P(x)}{Q(x)}$ polynomer.

Eksempel: $I = \int \frac{x^3 - x^2 - 4x - 5}{x^2 - 2x - 3}\,dx$

Polynomdivisjon:

$$
\begin{array}{l}
x^3 - x^2 - 4x - 5 : x^2 - 2x - 3 = \underline{x+1} \\
-(x^3 - 2x^2 - 3x) \\
\hline
\quad x^2 - x - 5 \\
\quad -(x^2 - 2x - 3) \\
\hline
\quad\quad x - 2
\end{array}
$$

dvs $\frac{x^3 - x^2 - 4x - 5}{x^2 - 2x - 3} = x + 1 + \frac{x-2}{x^2 - 2x - 3}$

Dermed $I = \int \left(x + 1 + \frac{x-2}{x^2-2x-3}\right) dx = \frac{x^2}{2} + x + \underbrace{\int \frac{x-2}{x^2-2x-3}\,dx}_{I_0}$

Faktorisere nevneren: $x^2 - 2x - 3 = 0$

$$x = \frac{2 \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot (-3)}}{2} = \frac{2 \pm \sqrt{16}}{2}$$

$$= \frac{2 \pm 4}{2} = \begin{cases} 3 \\ -1 \end{cases}$$

Altså $x^2 - 2x - 3 = (x-3)(x+1)$

Dermed: $I_0 = \int \frac{x-2}{(x-3)(x+1)}\,dx$

Delbrøker: $\frac{x-2}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1} \quad \Big| \cdot (x-3)(x+1)$ Må finne A og B

$$x - 2 = A(x+1) + B(x-3) = (A+B)x + A - 3B$$

Trenger: $A + B = 1$, $A - 3B = -2$ $\Rightarrow$

$$
\begin{array}{l}
A + B = 1 \\
-A + 3B = 2 \\
\hline
4B = 3 \Rightarrow B = \frac{3}{4}
\end{array}
$$

$A = \frac{1}{4}$

Tilbake til I_0: $I_0 = \int \frac{x-2}{(x-3)(x+1)}\,dx = \int \frac{\frac{1}{4}}{x-3}\,dx + \int \frac{\frac{3}{4}}{x+1}\,dx$

$$= \frac{1}{4}\ln|x-3| + \frac{3}{4}\ln|x+1| + C$$

Setter sammen:

$$I = \frac{x^2}{2} + x + I_0 = \frac{x^2}{2} + x + \frac{1}{4}\ln|x-3| + \frac{3}{4}\ln|x+1| + C$$

Hva skjer når å ikke kan faktorisere nevneren?

Minner om fullføring av kvadratet: $x^2+ax+b = \underbrace{x^2+ax+\left(\frac{a}{2}\right)^2}_{(x+\frac{a}{2})^2} - \left(\frac{a}{2}\right)^2 + b = \left(x+\frac{a}{2}\right)^2 - \frac{a^2}{4} + b$

Eksempel: $I = \int \frac{3}{x^2+4x+6}\,dx$

Prøver å faktorisere:

$x^2+4x+6=0$

$x = \frac{-4 \pm \sqrt{4^2-4\cdot 1\cdot 6}}{2} = \frac{-4\pm\sqrt{-8}}{2}$ ingen reell faktorisering

$$= \int \frac{3}{\underbrace{x^2+4x+2^2}_{(x+2)^2} \underbrace{-2^2+6}_{+\,2}}\,dx$$

$$= \int \frac{3}{(x+2)^2+2}\,dx$$

Husk at $(\arctan u)' = \frac{1}{u^2+1}$

$$= \int \frac{3}{2\left(\frac{(x+2)^2}{2}+1\right)}\,dx = \int \frac{3}{2\left(\left(\frac{x+2}{\sqrt{2}}\right)^2+1\right)}\,dx$$

$u = \frac{x+2}{\sqrt{2}}$

$du = \frac{1}{\sqrt{2}}\,dx$

$dx = \sqrt{2}\,du$

$$= \int \frac{3}{2(u^2+1)}\sqrt{2}\,du = \frac{3\sqrt{2}}{2}\int \frac{du}{1+u^2}$$

$$= \frac{3\sqrt{2}}{2}\arctan u + C = \frac{3\sqrt{2}}{2}\arctan\frac{x+2}{\sqrt{2}} + C$$

Eksempel: $I = \int \frac{x-3}{x^2+3x+3}\,dx$

Nævneren derived: $u = x^2+3x+3$, $u' = 2x+3$

Plan: smugle denne inn i telleren

$= \int \frac{1}{2}\frac{(2x-6)}{x^2+3x+3}\,dx$

$= \frac{1}{2}\int \frac{(2x+3)\overbrace{-3-6}^{-9}}{x^2+3x+3}\,dx = \frac{1}{2}\int \frac{\overbrace{2x+3}^{du}}{\underbrace{x^2+3x+3}_{u}}\,dx - \frac{9}{2}\int \frac{1}{x^2+3x+3}\,dx$

$u = x^2+3x+3$

$du = (2x+3)\,dx$

$= \frac{1}{2}\int \frac{1}{u}\,du - \frac{9}{2}\int \frac{1}{x^2+3x+3}\,dx$

$= \frac{1}{2}\ln u - \frac{9}{2}\int \frac{1}{x^2+3x+3}\,dx = \frac{1}{2}\ln(x^2+3x+3) - \frac{9}{2}\underbrace{\int \frac{1}{x^2+3x+3}\,dx}_{I_0}$

Regner ut I_0:

$I_0 = \int \frac{1}{x^2+3x+3}\,dx = \int \frac{1}{\underbrace{x^2+3x+\left(\frac{3}{2}\right)^2}_{(x+\frac{3}{2})^2} \underbrace{-\left(\frac{3}{2}\right)^2+3}_{-\frac{9}{4}+\frac{12}{4} = 3/4}}\,dx$

$\boxed{\frac{1}{u^2+1}}$

$= \int \frac{1}{\left(x+\frac{3}{2}\right)^2+\frac{3}{4}}\,dx = \int \frac{1\cdot\frac{4}{3}}{\frac{4}{3}\cdot\frac{3}{4}\left(\frac{4}{3}\left(x+\frac{3}{2}\right)^2+1\right)}\,dx$

$\frac{4}{3}\left(x+\frac{3}{2}\right)^2 = \left(\frac{2x+3}{\sqrt{3}}\right)^2$

$= \frac{4}{3}\int \frac{1}{\left(\frac{2x+3}{\sqrt{3}}\right)^2+1}\,dx$

$u = \frac{2x+3}{\sqrt{3}}$

$du = \frac{2}{\sqrt{3}}\,dx$

$dx = \frac{\sqrt{3}}{2}\,du$

$= \frac{4}{3}\int \frac{\frac{\sqrt{3}}{2}}{u^2+1}\,du$

$= \frac{\overset{2}{\cancel{4}}}{3}\cdot\frac{\sqrt{3}}{\cancel{2}}\arctan u + C = \frac{2\sqrt{3}}{3}\arctan\frac{2x+3}{\sqrt{3}} + C$

Derved:

$I = \frac{1}{2}\ln(x^2+3x+3) - \frac{\overset{3}{\cancel{9}}}{\cancel{2}}\cdot\frac{\cancel{2}\sqrt{3}}{\cancel{3}}\arctan\frac{2x+3}{\sqrt{3}} + C$

$= \frac{1}{2}\ln(x^2+3x+3) - 3\sqrt{3}\arctan\frac{2x+3}{\sqrt{3}} + C$

Eksempel: $I = \int \frac{5x^2+3x+7}{(x-1)(x^2+2x+2)}\,dx$

Delbrøksoppspalting: kan vi finne konstanter A, B, C slik at

$$\frac{5x^2+3x+7}{(x-1)(x^2+2x+2)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+2x+2} \quad \cdot (x-1)(x^2+2x+2)$$

$$5x^2+3x+7 = A(x^2+2x+2) + (Bx+C)(x-1)$$
$$= Ax^2 + 2Ax + 2A + Bx^2 - Bx + Cx - C$$
$$= (A+B)x^2 + (2A-B+C)x + 2A-C$$

Dette gjør hvis

(I) $A+B=5$ (II) $2A-B+C=3$ (III) $2A-C=7$

Adderer (II) + (III): $4A - B = 10$
$A + B = 5$ } Legger sammen

$5A = 15 \Rightarrow A=3,\ B=2,\ C=-1$

Altså
$$\frac{5x^2+3x+7}{(x-1)(x^2+2x+2)} = \frac{3}{x-1} + \frac{2x-1}{x^2+2x+2}$$

Dermed
$$\int \frac{5x^2+3x+7}{(x-1)(x^2+2x+2)}\,dx = \int \frac{3}{x-1}\,dx + \int \frac{2x-1}{x^2+2x+2}\,dx$$
$$= 3\ln|x-1| + \underbrace{\int \frac{2x-1}{x^2+2x+2}\,dx}_{I_0}$$

Løser I_0:

Vennen: $u = x^2+2x+2$
$u' = 2x+2$
smugler inn i telleren.

$$I = \int \frac{2x-1}{x^2+2x+2}\,dx$$
$$= \int \frac{(2x+2)\overbrace{-2-1}^{-3}}{x^2+2x+2}\,dx = \int \frac{2x+2}{x^2+2x+2}\,dx - 3\int \frac{1}{x^2+2x+2}\,dx$$

$u = x^2+2x+2$
$du = (2x+2)dx$

$$= \int \frac{du}{u} - 3\int \frac{1}{x^2+2x+2}\,dx$$
$$= \ln u - 3\int \frac{1}{\underbrace{x^2+2x+1}_{(x+1)^2}\underbrace{-1+2}_{+1}}\,dx$$

$u = x+1$
$du = dx$

$$= \ln(x^2+2x+2) - 3\int \frac{1}{(x+1)^2+1}\,dx$$
$$= \ln(x^2+2x+2) - 3\int \frac{1}{u^2+1}\,du = \ln(x^2+2x+2) - 3\arctan u + C$$
$$= \ln(x^2+2x+2) - 3\arctan(x+1) + C$$

Generell mönster

$$\int \frac{P(x)}{Q(x)}\,dx \qquad \text{grad } P(x) \leq \text{grad } Q(x)$$

Algebraens fundamentalteorem:

$$Q(x) = (x-r_1)\ldots(x-r_k)(x^2+a_1x+b_1)\ldots(x^2+a_qx+b_q)$$

$$= (x-r_1)^{m_1}\ldots(x-r_2)^{m_2}(x^2+a_1x+b_1)^{n_1}\ldots(x^2+a_px+b_p)^{n_p}$$

Delbrøksoppspalting:

$$\frac{P(x)}{Q(x)} = \underbrace{\frac{A_1}{x-r_1} + \frac{A_2}{(x-r_1)^2} + \ldots + \frac{A_{m_1}}{(x-r_1)^{m_1}}}_{\text{ledd som kommer fra } (x-r_1)^{m_1}}$$

$$+ \text{ ledd som kommer fra } (x-r_2)^{m_2} + \ldots$$

$$\vdots$$

$$+ \underbrace{\frac{B_1x+C_1}{(x^2+a_1x+b_1)} + \ldots + \frac{B_{n_1}x+C_{n_1}}{(x^2+a_1x+b_1)^{n_1}}}$$ ledd som kommer fra $(x^2+a_1x+b_1)^{n_1}$

$$+ \ldots$$

Eksempel:

$$\frac{7x^7+3x^4+2x-4}{\underbrace{(x-1)}_{1}\underbrace{(x+2)^3}_{3.}\underbrace{(x^2+4x+5)^2}_{4.}}$$

7. grad

8. grad

$(x-1)(x+2)^3(x^2+4x+5)^2$

$$= \frac{A}{(x-1)} + \frac{B}{x+2} + \frac{C}{(x+2)^2} + \frac{D}{(x+2)^3} + \frac{Ex+F}{x^2+4x+5} + \frac{Gx+H}{(x^2+4x+5)^2}$$

8 ligninger med 8 ukjente.

Eksempel:
$$\frac{4x^3-11x^2+18x-4}{(x-2)^2(x^2+2x+2)} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{Cx+D}{x^2+2x+2} \quad \Big|$$

ganger med $(x-2)^2(x^2+2x+2)$

$$4x^3-11x^2+18x-4 = A(x-2)(x^2+2x+2) + B(x^2+2x+2) + (Cx+D)(x-2)^2$$

$$= (A+C)x^3 + (B-4C+D)x^2 + (-2A+2B-4C-4D)x + (-2A+2B+4D)$$

$A+C=4$, $B-4C+D=-11$, $-2A+2B-4C-4D=18$, $-2A+2B+4D=-4$

4 ligninger / 4 ukjente. $A=1$, $B=2$, $C=3$, $D=-1$.