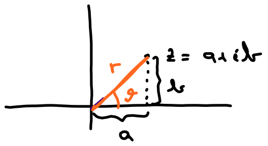
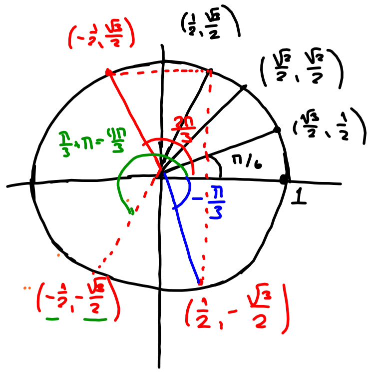
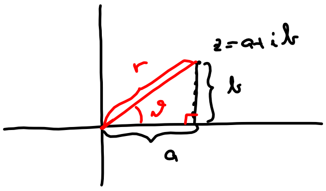


Trigonometri:

θ	$\sin\theta$	$\cos\theta$
0	0	1
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{\pi}{2}$	1	0



Kartesiske koordinater / polarkoordinater



Oversette fra polarkoordinater til kartesiske:

$a = r\cos\theta$, $b = r\sin\theta$

Oversette fra kartesiske koor. til polarkoordinater

$r = \sqrt{a^2 + b^2}$

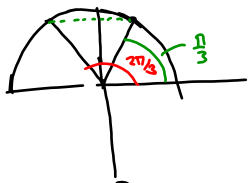
$\cos\theta = \frac{a}{r}$, $\sin\theta = \frac{b}{r}$

bestemmer θ

Eksempel: Finn a og b når $r = 4$ og $\theta = \frac{2\pi}{3}$ ($= 120°$)

$a = r\cos\theta = 4\cos\frac{2\pi}{3} = 4\left(-\frac{1}{2}\right) = -2$

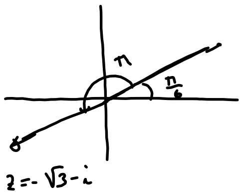
$b = r\sin\theta = 4\sin\frac{2\pi}{3} = 4 \cdot \frac{\sqrt{3}}{2} = 2\sqrt{3}$



$z = a + ib = \underline{-2 + i2\sqrt{3}}$

Eksempel: Finn r og θ når $z = -\sqrt{3} - i$
$a = -\sqrt{3}$
$b = -1$

$r = \sqrt{a^2 + b^2} = \sqrt{(-\sqrt{3})^2 + (-1)^2} = \sqrt{3 + 1} = \underline{\underline{2}}$



$\sin\theta = \frac{b}{r} = \frac{-1}{2} = \underline{\underline{-\frac{1}{2}}}$

Siden vinkelen ligger i 3. kvadrant og

$\sin\theta = -\frac{1}{2}$, må $\theta = \pi + \frac{\pi}{6} = \underline{\underline{\frac{7\pi}{6}}}$

Eksempel: Anta at z_1 og z_2 er to komplekse tall med polarkoordinater:

z_1: $r_1 = 3$, $\theta_1 = \frac{7\pi}{12}$, z_2: $r_2 = 4$, $\theta_2 = \frac{8\pi}{12}$

Regn ut: $z = z_1 z_2$

$r = r_1 r_2 = 3 \cdot 4 = 12$

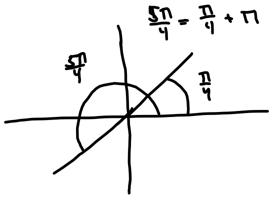
Geometrisk tolkning av multiplikasjon: z $\theta = \theta_1 + \theta_2 = \frac{7\pi}{12} + \frac{8\pi}{12} = \frac{15\pi}{12} = \frac{5\pi}{4}$

$\frac{5\pi}{4} = \frac{\pi}{4} + \pi$

$z = r\cos\theta + ir\sin\theta = 12\cos\frac{5\pi}{4} + i\,12\sin\frac{5\pi}{4}$

$= 12\left(-\frac{\sqrt{2}}{2}\right) + i\,12\left(-\frac{\sqrt{2}}{2}\right)$

$= \underline{\underline{-6\sqrt{2} - 6i\sqrt{2}}}$

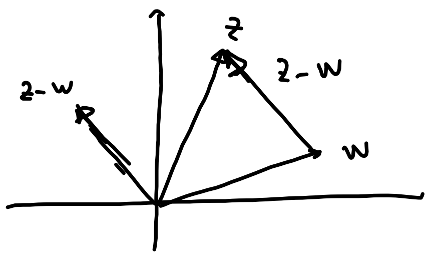


Mer geometri i det komplekse planet

Tallverdi: $|z| = \sqrt{a^2+b^2} = r$

Observasjon: $z \cdot \bar{z} = (a+ib)(a-ib) = a^2 \underbrace{-(ib)^2}_{+b^2} = a^2+b^2 = |z|^2$

dvs: $|z|^2 = z \cdot \bar{z}$.

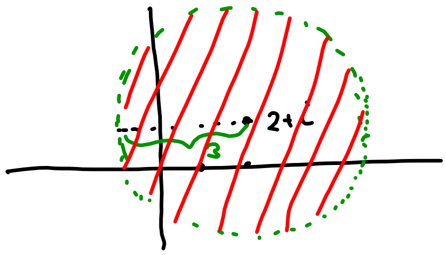


$w + (z-w) = z$

$|z-w|$ = avstanden mellom z og w.

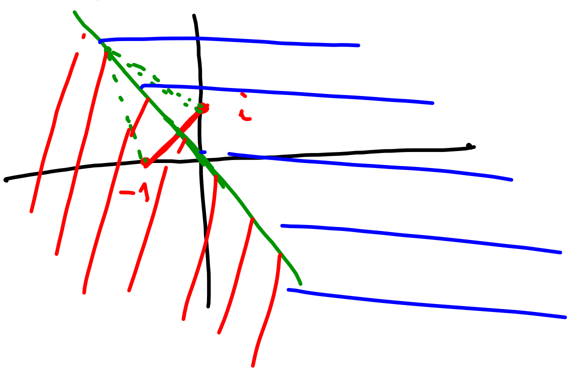
Eksempel: Finn de punktene z i det komplekse planet slik at

$\underbrace{|z-(2+i)|}_{\text{avstanden mellom } z \text{ og } 2+i} < 3$



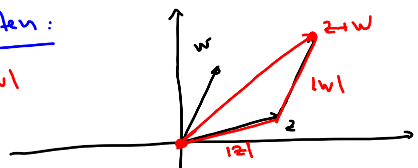
Eksempel: Finn de punktene z i det komplekse planet slik at

$\underbrace{|z-i|}_{\text{avstanden fra } z \text{ til } i} \leq |z+1| = \underbrace{|z-(-1)|}_{\text{avstanden fra } z \text{ til } (-1)}$



Trekantulikheten:

$|z+w| \le |z|+|w|$



$|z_1+z_2+\dots+z_n| \le |z_1|+|z_2|+\dots+|z_n|$

$$\boxed{(e^n)^z = (e^z)^n}$$

Eksponentiell form

Vi vet hva e^a er når a er et reelt tall.

Men hva skal e^z bety når z er et komplekst tall $z=a+ib$?

Definisjon: For et komplekst tall $z=a+ib$ definerer vi

$$e^z = e^{a+ib} = e^a(\cos b + i\sin b)$$ = det komplekse tallet med ~~modulus argument~~ modulus e^a og argument b.

Noen verdier: $z=a=a+i0$

$$e^z = e^a(\underbrace{\cos 0}_{1} + i\underbrace{\sin 0}_{0}) = e^a$$

$z=i\pi = 0+i\pi$

$$e^{i\pi} = e^z = \underbrace{e^0}_{1}(\underbrace{\cos\pi}_{-1} + i\underbrace{\sin\pi}_{0}) = -1$$

$$\boxed{e^{i\pi}+1=0}$$ Eulers formel

Teorem: Hvis z og w er komplekse tall

$$e^{z+w} = e^z e^w$$

Bevis: La $z=a+ib$, $w=c+id$. Regner ut de to uttrykkene:

$$e^{z+w} = e^{(a+c)+i(b+d)} = e^{a+c}(\cos(b+d)+i\sin(b+d))$$

$e^z \cdot e^w$ $\left(e^z = e^{a+ib} = e^a(\cos b + i\sin b)\right.$ modulus e^a / argument b $)$

$\left(e^w = e^{c+id} = e^c(\cos d + i\sin d)\right.$ modulus e^c argument d $)$

$e^z \cdot e^w$ modulus $e^a \cdot e^c = e^{a+c}$

argument $b+d$

$$e^z\cdot e^w = e^{a+c}(\cos(b+d)+i\sin(b+d))$$

Altså er $e^{z+w} = e^z\cdot e^w$

Utvidelse: $e^{z_1+\dots+z_n} = e^{z_1}\cdot e^{z_2}\cdots e^{z_n}$

Hvorfor: $e^{(z_1+\dots+z_{n-1})+z_n} = e^{z_1+\dots+z_{n-1}}\cdot e^{z_n}$

$= e^{(z_1+\dots+z_{n-2})+z_{n-1}}\cdot e^{z_n} = e^{z_1+\dots+z_{n-2}}\cdot e^{z_{n-1}}\cdot e^{z_n}$

Konsekvens: $e^{nz} = e^{z+z+\dots+z} = e^z\cdot e^z\cdots e^z = (e^z)^n$

Formler:

$e^{z+w} = e^z\cdot e^w$

$e^{nz} = (e^z)^n$

De Moivres formel: $(\cos\vartheta + i\sin\vartheta)^n = \cos n\vartheta + i\sin n\vartheta$

Bevis: $e^{i\vartheta} = e^{0+i\vartheta} = e^0(\cos\vartheta + i\sin\vartheta) = \cos\vartheta + i\sin\vartheta$

$$(\cos\vartheta + i\sin\vartheta)^n = (e^{i\vartheta})^n = e^{in\vartheta} = \cos n\vartheta + i\sin n\vartheta$$

Eksempel: Regn ut $(1+i)^{37}$.

Skriv $1+i$ på polarform: $r = \sqrt{a^2+b^2} = \sqrt{1^2+1^2} = \sqrt2 = 2^{1/2}$

$\sin\vartheta = \frac{b}{r} = \frac{1}{\sqrt2} = \frac{\sqrt2}{2}$, siden ϑ ligger i første kvadrant, er $\vartheta = \frac{\pi}{4}$.

$\frac{37}{2} = 18 + \frac12$

$$1+i = r(\cos\vartheta + i\sin\vartheta) = 2^{1/2}\left(\cos\frac\pi4 + i\sin\frac\pi4\right)$$

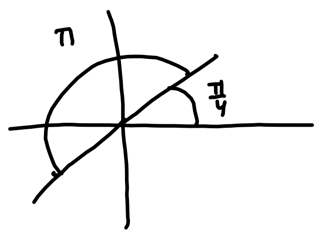
Dermed $(1+i)^{37} = \left[2^{1/2}\left(\cos\frac\pi4 + i\sin\frac\pi4\right)\right]^{37} = 2^{37/2}\left(\cos\frac\pi4 + i\sin\frac\pi4\right)^{37}$

$$= 2^{18}\cdot\sqrt2\left(\cos\frac{37\pi}{4} + i\sin\frac{37\pi}{4}\right)$$

$$\boxed{\frac{37}{4}\pi = \underbrace{8\pi}_{\frac{32}{4}\pi} + \frac54\pi}$$

$$= 2^{18}\sqrt2\left(\cos\left(\frac54\pi + 8\pi\right) + i\sin\left(\frac54\pi + 8\pi\right)\right)$$

$$= 2^{18}\sqrt2\left(\underbrace{\cos\frac54\pi}_{-\frac{\sqrt2}{2}} + i\underbrace{\sin\frac54\pi}_{-\frac{\sqrt2}{2}}\right)$$



$$= 2^{18}\sqrt2\left(-\frac{\sqrt2}{2} - i\frac{\sqrt2}{2}\right) = 2^{18}(-1-i)$$