Delbrøksoppspalting

$$\int \frac{P(x)}{Q(x)}\,dx = \int \frac{A_1}{(x-r_1)}\,dx + \dots + \int \frac{A_n}{(x-r_1)^n} +$$

$$\dots + \int \frac{B_1x+C_1}{x^2+a_1x+b_1}\,dx + \dots + \int \frac{B_mx+C_m}{(x^2+a_1x+b_1)^m}\,dx$$

$$+ \dots$$

Løsninger:

$$\int \frac{A_1}{(x-r_1)}\,dx = A_1 \ln|x-r_1| + C$$

$$\int \frac{A_n}{(x-r_1)^n}\,dx = \int A_n (x-r_1)^{-n}\,dx$$

$$= A_n \frac{(x-r_1)^{-n+1}}{-n+1} + C = \frac{A_n}{1-n} \cdot \frac{1}{(x-r_1)^{n-1}} + C$$

$\int \frac{B_1x+C_1}{x^2+a_1x+b_1}\,dx$ .... lærte vi i går.

$\int \frac{B_1x+C_1}{(x^2+a_1x+b_1)^m}\,dx$ ..... les i boka hvis du tør!

Eksempel: $\int \frac{\arctan x}{(x-1)^2}\,dx$

$u = \arctan x$, $v' = \frac{1}{(x-1)^2} = (x-1)^{-2}$

$u' = \frac{1}{1+x^2}$, $v = -\frac{1}{(x-1)}$

$$= -\frac{1}{(x-1)}\arctan x + \int \frac{1}{1+x^2} \cdot \frac{1}{(x-1)}\,dx$$

$$= -\frac{\arctan x}{(x-1)} + \int \frac{1}{(x^2+1)(x-1)}\,dx$$

Delbrøksoppspalting: $\frac{1}{(x^2+1)(x-1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1} \quad \Big| \, (x^2+1)(x-1)$

$$1 = A(x^2+1) + (Bx+C)(x-1) = Ax^2 + A + Bx^2 - Bx + Cx - C$$

$$= (A+B)x^2 + (-B+C)x + A - C$$

$A+B=0$, $-B+C=0$, $A-C=1$

$A=-B$, $C=B$, $-B-B=1 \Rightarrow B=-\frac{1}{2}$

$A=\frac{1}{2}$

$C=-\frac{1}{2}$

Altså

$$\int \frac{1}{(x^2+1)(x-1)}\,dx = \int \frac{\frac{1}{2}}{x-1}\,dx + \int \frac{-\frac{1}{2}x - \frac{1}{2}}{x^2+1}\,dx$$

$$= \frac{1}{2}\ln|x-1| - \frac{1}{2}\int \frac{x+1}{x^2+1}\,dx$$

$u = x^2+1$

$u' = 2x$

$$= \frac{1}{2}\ln|x-1| - \frac{1}{2}\int \frac{\frac{1}{2}(2x+2)}{x^2+1}\,dx$$

$$= \frac{1}{2}\ln|x-1| - \frac{1}{4}\int \left(\frac{2x}{x^2+1} + \frac{2}{x^2+1}\right)dx$$

$du = 2x\,dx$

$$= \frac{1}{2}\ln|x-1| - \frac{1}{4}\int \frac{du}{u} - \frac{1}{2}\int \frac{1}{x^2+1}\,dx$$

$$= \frac{1}{2}\ln|x-1| - \frac{1}{4}\ln|u| - \frac{1}{2}\arctan x + C$$

$$= \frac{1}{2}\ln|x-1| - \frac{1}{4}\ln(x^2+1) - \frac{1}{2}\arctan x + C$$

Dette gir

$$\int \frac{\arctan x}{(x-1)^2}\,dx = -\frac{\arctan x}{(x-1)} + \frac{1}{2}\ln|x-1| - \frac{1}{4}\ln(x^2+1) - \frac{1}{2}\arctan x + C$$

Integraler av typen $\int \sin^n x \cos^m x \, dx$ (9.4)

Enten n eller m er odde:

$\cos^2 x = 1 - \sin^2 x$

$u = \sin x$

$du = \cos x \, dx$

Eksempel: $\displaystyle\int \sin^2 x \cos^3 x \, dx = \int \sin^2 x \underbrace{\cos^2 x}_{(1-\sin^2 x)} \overbrace{\cos x \, dx}^{du}$

$$= \int u^2 (1-u^2) \, du = \int (u^2 - u^4) \, du = \frac{u^3}{3} - \frac{u^5}{5} + C$$

$$= \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C$$

Hvis både n og m er like:

$$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$

$$\cos^2 x = \frac{1 + \cos 2x}{2}, \quad \sin^2 x = \frac{1 - \cos 2x}{2}$$

Eksempel: $\displaystyle\int \sin^2 x \cos^2 x \, dx = \int \frac{1-\cos 2x}{2} \cdot \frac{1+\cos 2x}{2} \, dx$

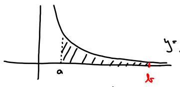
$$= \frac{1}{4} \int (1 - \cos^2 2x) \, dx = \frac{x}{4} - \frac{1}{4} \int \cos^2 2x \, dx$$

$$= \frac{x}{4} - \frac{1}{4} \int \frac{1 + \cos 4x}{2} \, dx = \frac{x}{4} - \frac{1}{8} \int (1 + \cos 4x) \, dx$$

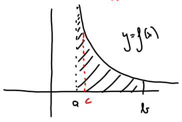
$$= \frac{x}{4} - \frac{x}{8} - \frac{1}{8} \int \cos 4x \, dx = \frac{x}{8} - \frac{1}{8} \frac{\sin 4x}{4} + C$$

$$= \frac{x}{8} - \frac{1}{32} \sin 4x + C$$

Uegentlige integraler



$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx$$



$$\int_a^b f(x)\,dx = \lim_{c\to a^+}\int_c^b f(x)\,dx$$

Definisjon: Anta at $f: [a,\infty)\to\mathbb{R}$ er en kontinuerlig funksjon. Vi sier at det uegentlige integralet $\int_a^\infty f(x)\,dx$ konvergerer dersom

$$\lim_{b\to\infty}\int_a^b f(x)\,dx$$

eksisterer (som et tall). Hvis ikke, sier vi at integralet divergerer.

Hvis integralet konvergerer, setter vi

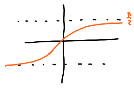
$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx.$$

Eksempel: $\int_0^\infty \frac{1}{1+x^2}\,dx$. Divergens/konvergens verdi?

$$\lim_{b\to\infty}\int_0^b \frac{1}{1+x^2}\,dx = \lim_{b\to\infty}\left[\arctan x\right]_0^b = \lim_{b\to\infty}\left[\arctan b - \cancel{\arctan 0}\right] = \frac{\pi}{2}$$

Integralet konverger og

$$\int_0^\infty \frac{1}{1+x^2}\,dx = \frac{\pi}{2}$$



Eksempel: $\int_1^\infty \frac{1}{x}\,dx$

$$\lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty}\left[\ln x\right]_1^b = \lim_{b\to\infty}\left[\ln b - \ln 1\right] = \infty$$

Integralet divergerer.

Teorem: Integralet $\int_1^\infty \frac{1}{x^p}\,dx$ konvergerer for $p>1$ og divergerer for $p\le 1$.

Bevis: Vet allerede at vi har divergens for $p=1$. Hvis $p\neq 1$, har vi

$$\lim_{b\to\infty}\int_1^b \frac{1}{x^p}\,dx = \lim_{b\to\infty}\int_1^b x^{-p}\,dx = \lim_{b\to\infty}\left[\frac{x^{-p+1}}{-p+1}\right]_1^b$$

$$= \lim_{b\to\infty}\left[\frac{b^{-p+1}}{1-p} - \frac{1}{1-p}\right]$$

$b^{-p+1} \to \infty$ hvis $\underbrace{-p+1>0}_{p<1}$

$b^{-p+1} \to 0$ hvis $-p+1<0$, $p>1$

konvergens for $p>1$
divergens for $p<1$ p divergens

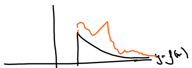
Spesielt: $p=2$
$$\lim_{b\to\infty}\int_1^b \frac{1}{x^2}\,dx = \lim_{b\to\infty}\left[-\frac{1}{x}\right]_1^b = \lim_{b\to\infty}\left[-\frac{1}{b}+\frac{1}{1}\right] = 1$$

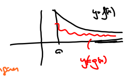
$$\int_1^\infty \frac{1}{x^2}\,dx = 1$$

Sammenligningstesten: $f,g: [a,\infty)\to\mathbb{R}$, kontinuerlige, $f(x), g(x)\ge 0$.

(i) Hvis $\int_a^\infty f(x)\,dx$ konvergerer og $0\le g(x)\le f(x)$, så konvergerer også $\int_a^\infty g(x)\,dx$.

(ii) Hvis $\int_a^\infty f(x)\,dx$ divergerer og $0\le f(x)\le g(x)$, så divergerer også $\int_a^\infty g(x)\,dx$





Eksempel: Vis at $\int_1^\infty \frac{1}{x^2+2x+4}\,dx$ konvergerer

$$0 \le \frac{1}{x^2+2x+4} \le \frac{1}{x^2}$$

Vet at $\int_1^\infty \frac{1}{x^2}\,dx$ konvergerer, så $\int_1^\infty \frac{1}{x^2+2x+4}\,dx$ vil også konvergere.

Grensesammenligningstesten: Anta at $f,g: [a,\infty)\to\mathbb{R}$ er positive og kontinuerlige.

(i) Hvis $\int_a^\infty f(x)\,dx$ konvergerer og $\lim_{x\to\infty}\frac{g(x)}{f(x)} < \infty$, så konvergerer $\int_a^\infty g(x)\,dx$ også.

(ii) Hvis $\int_a^\infty f(x)\,dx$ divergerer og $\lim_{x\to\infty}\frac{g(x)}{f(x)} > 0$, så divergerer også $\int_a^\infty g(x)\,dx$.

Eksempel: Undersøk om $\int_1^\infty \frac{x+4}{x^2+2x+6}\,dx$ divergerer eller konvergerer.

$$\approx \frac{x}{x^2} = \frac{1}{x}$$

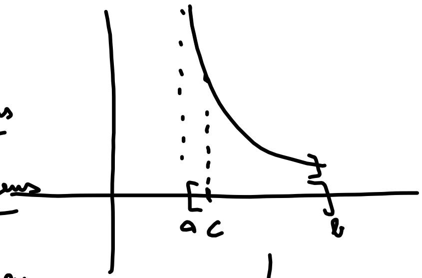
Bruker grensesammenligningstesten med $\underbrace{f(x)=\frac{1}{x}}_{\text{divergerer}}$ og $g(x)=\frac{x+4}{x^2+2x+6}$

$$\lim_{x\to\infty}\frac{g(x)}{f(x)} = \lim_{x\to\infty}\frac{\frac{x+4}{x^2+2x+6}}{\frac{1}{x}} = \lim_{x\to\infty}\frac{x^2+4x}{x^2+2x+6} = \lim_{x\to\infty}\frac{\cancel{x^2}(1+\frac{4}{x})}{\cancel{x^2}(1+\frac{2}{x}+\frac{6}{x^2})}$$

$$= \lim_{x\to\infty}\frac{1+\frac{4}{x}}{1+\frac{2}{x}+\frac{6}{x^2}} = \frac{1}{1} = 1 > 0$$

Konklusjon: $\int_1^\infty \frac{x+4}{x^2+2x+6}\,dx$ divergerer.

Andre type uegentlig integral

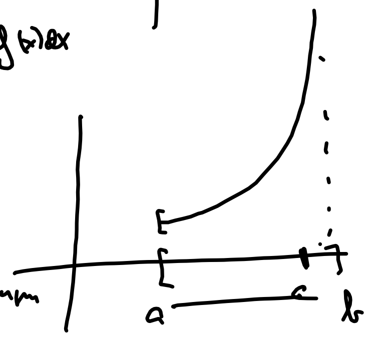


$f: (a,b] \to \mathbb{R}$:

$\lim_{c \to a^+} \int_c^b f(x)\,dx$ → endelig, så konvergens

→ uendelig, så divergens

Hvis konvergens, så $\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx$



$f: [a,b) \to \mathbb{R}$

$\lim_{c \to b^-} \int_a^c f(x)\,dx$ → endelig, så konvergens

→ uendelig, så divergens

Hvis konvergens, så $\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx$.

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