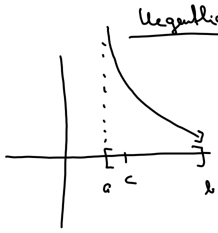
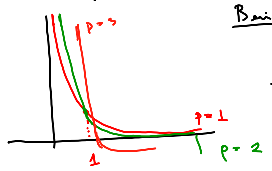
Uegentlige integraler



$$\int_a^b f(x)\,dx = \lim_{c\to a^+}\int_c^b f(x)\,dx$$ forutsatt at integralet konvergerer.

Teorem: Integralet $\int_0^1 \frac{1}{x^p}\,dx$ divergerer for $p \geq 1$ og konvergerer for $p<1$.



Bevis: $p=1$: $\int_0^1 \frac{1}{x}\,dx = \lim_{c\to 0^+}\int_c^1 \frac{1}{x}\,dx$

$$= \lim_{c\to 0^+}\left[\ln x\right]_c^1 = \lim_{c\to 0^+}\left[0-\underset{\downarrow\,-\infty}{\ln c}\right] = \infty$$

Integralet divergerer.

$p \neq 1$: $$\int_0^1 \frac{1}{x^p}\,dx = \lim_{c\to 0^+}\int_c^1 x^{-p}\,dx = \lim_{c\to 0^+}\left[\frac{x^{-p+1}}{-p+1}\right]_c^1$$

$$= \lim_{c\to 0^+}\left[\frac{1}{-p+1} - \frac{c^{-p+1}}{-p+1}\right]$$

$\to 0$ hvis $-p+1>0$ dvs $p<1$.

∞ hvis $-p+1<0$ dvs $p>1$

Konklusjon: Divergerer for $p \geq 1$
konvergerer for $p<1$.

Doble grenser:

$$\int_{-\infty}^a f(x)\,dx = \lim_{c\to -\infty}\int_c^a f(x)\,dx$$ konvergerer hvis grenseverdien er endelig, divergerer.

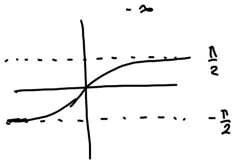
Integralet $\int_{-\infty}^{\infty} f(x)\,dx$ konvergerer dersom $\int_{-\infty}^a f(x)\,dx$ og $\int_a^{\infty} f(x)\,dx$ begge konvergerer, og i så fall er

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^{\infty} f(x)\,dx$$ (uavhengig av hvilken a jeg bruker).

Eksempel: $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$

Sjekker: $$\int_{-\infty}^0 \frac{1}{1+x^2}\,dx = \lim_{c\to -\infty}\int_c^0 \frac{1}{1+x^2}\,dx = \lim_{c\to -\infty}\left[\arctan x\right]_c^0$$

$$= \lim_{c\to -\infty}\left[-\underbrace{\arctan c}_{-\frac{\pi}{2}}\right] = \frac{\pi}{2}.$$



$$\int_0^{\infty} \frac{1}{1+x^2}\,dx = \lim_{c\to\infty}\int_0^c \frac{1}{1+x^2}\,dx = \lim_{c\to\infty}\left[\arctan x\right]_0^c$$

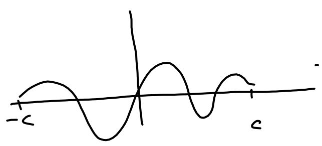
$$= \lim_{c\to\infty} \arctan c = \frac{\pi}{2}$$

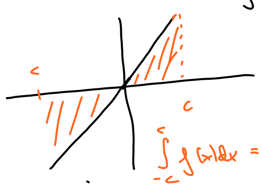
Siden både $\int_{-\infty}^0 \frac{1}{1+x^2}\,dx$ og $\int_0^{\infty} \frac{1}{1+x^2}\,dx$ konvergerer, så konvergerer også

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_0^{\infty} \frac{1}{1+x^2}\,dx = \frac{\pi}{2}+\frac{\pi}{2} = \pi$$

Hvorfor definerer vi ikke $\int_{-\infty}^{\infty} f(x)\,dx = \lim_{c\to\infty}\int_{-c}^{c} f(x)\,dx$?

$f(x) = x$





$$\int_{-c}^{c} f(x)\,dx = 0$$

Dersom vi vet at $\int_{-\infty}^{\infty} f(x)\,dx$ konvergerer, så er

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{c\to\infty}\int_{-c}^{c} f(x)\,dx.$$

n-tupler (FVLA kap 1.1)

Def: Et n-tuppel av reelle tall er bare en liste $(a_1, a_2, \ldots, a_n)$ av n reelle tall.

Eksempel: $(-1, 2, \sqrt{3})$ - tre-tuppel, tripler.
$(\pi, \frac{2}{\sqrt{17}}, -\frac{2}{3}, 4, \pi^2)$ fem-tuppel.

Hvis vi har to n-tupler $\vec{a} = (a_1, a_2, \ldots, a_n)$, $\vec{b} = (b_1, b_2, \ldots, b_n)$, så definerer vi $\vec{a} + \vec{b} = (a_1+b_1, a_2+b_2, \ldots, a_n+b_n)$.

Eksempel: $\vec{a} = (1, -4, 3, 2)$, $\vec{b} = (2, -3, 0, 1)$
$\vec{a} + \vec{b} = (3, -7, 3, 3)$

Tilsvarende: $\vec{a} - \vec{b} = (a_1 - b_1, a_2 - b_2, \ldots, a_n - b_n)$

Multiplikasjon med tall (skalar):
$s \cdot \vec{a} = s(a_1, a_2, \ldots, a_n) = (sa_1, sa_2, \ldots, sa_n)$

Skalarprodukt / prikkprodukt:
$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + \ldots + a_n b_n$ (OBS. et tall)

Nullvektor: $\vec{0} = (0, 0, \ldots, 0)$
$-\vec{a} = (-a_1, -a_2, \ldots, -a_n)$

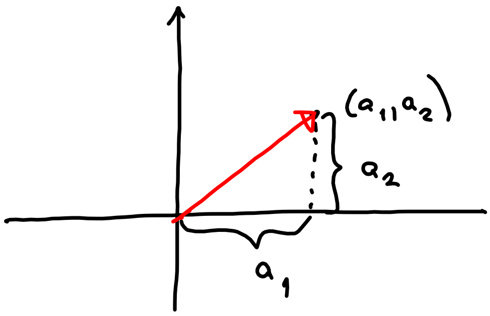
$\mathbb{R}^n = \{(a_1, \ldots, a_n) : a_1, a_2, \ldots, a_n \in \mathbb{R}\}$ = samlingen av alle reelle n-tupler.

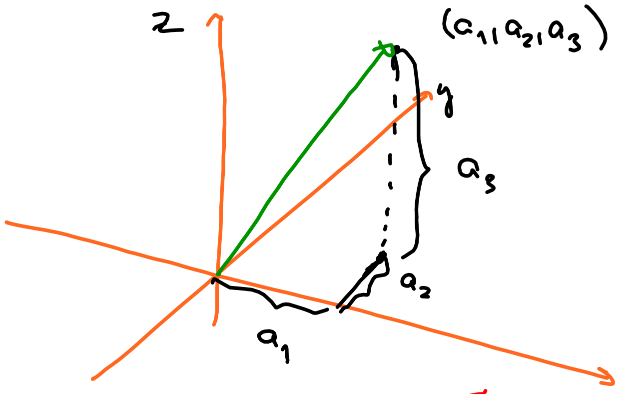
Eksempler: Firmaet ditt har n medarbeidere.
Lønn: $\vec{l} = (l_1, l_2, \ldots, l_n)$
7% tillegg: $\vec{l}_{ny} = 1.07\vec{l} = (1.07 l_1, \ldots, 1.07 l_n)$
Antall timer: $\vec{t} = (t_1, t_2, \ldots, t_n)$
Utbetaling: $t_1 l_1 + t_2 l_2 + \ldots + t_n l_n = \vec{t} \cdot \vec{l}$

Regneregler:
$\left.\begin{array}{l} \vec{a} + \vec{b} = \vec{b} + \vec{a} \\ \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a} \end{array}\right\}$ kommutative lover

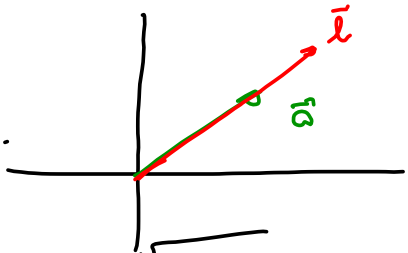
$\left.\begin{array}{l} \vec{a}(\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} \\ s(\vec{b} + \vec{c}) = s\vec{b} + s\vec{c} \end{array}\right\}$ distributive

Geometri for n-tupler





I $n=2,3$:



$\vec{a} \parallel \vec{b} \Leftrightarrow \vec{a} = t\vec{b}$ for en $t \neq 0$.

$\vec{a} \perp \vec{b} \Leftrightarrow \vec{a} \cdot \vec{b} = 0$

Lengden til $\vec{a}$: $n=2$: $\vec{a} = (a_1, a_2)$, $|\vec{a}| = \sqrt{a_1^2 + a_2^2}$

$n=3$: $\vec{a} = (a_1, a_2, a_3)$, $|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

Definisjon: Vi sier at to n-tupler $\vec{a} = (a_1, a_2, \ldots, a_n)$ og $\vec{b} = (b_1, b_2, \ldots, b_n)$ er parallelle dersom det finnes et tall t slik $\vec{a} = t\vec{b}$.

Vi sier at $\vec{a}$ og $\vec{b}$ er ortogonale / står normalt på hverandre dersom $\vec{a} \cdot \vec{b} = 0$.

Vi definerer lengden / normen til $\vec{a} = (a_1, a_2, \ldots, a_n)$ ved $|\vec{a}| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$

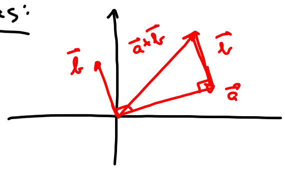
Bemerkning: $|\vec{a}|^2 = a_1^2 + a_2^2 + \cdots + a_n^2 = \vec{a} \cdot \vec{a}$

$$|t\vec{a}| = \sqrt{(ta_1)^2 + (ta_2)^2 + \cdots + (ta_n)^2}$$

$$= \sqrt{t^2 a_1^2 + t^2 a_2^2 + \cdots + t^2 a_n^2}$$

$$= \sqrt{|t|^2 (a_1^2 + a_2^2 + \cdots + a_n^2)} = |t||\vec{a}|$$

Pytagoras:



Pytagoras: $|\vec a+\vec b|^2 = |\vec a|^2+|\vec b|^2$, $\vec a \perp \vec b$

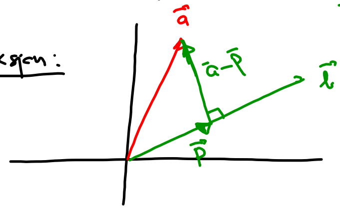
Pytagoras for n-tupler: Hvis n-tuplene $\vec a, \vec b$ står normalt på hverandre, så er

$$|\vec a+\vec b|^2 = |\vec a|^2+|\vec b|^2$$

Bevis: $|\vec a+\vec b|^2 = (\vec a+\vec b)\cdot(\vec a+\vec b) = \underbrace{\vec a\cdot\vec a}_{|\vec a|^2} + \underbrace{\vec a\cdot\vec b}_{0} + \underbrace{\vec b\cdot\vec a}_{0} + \underbrace{\vec b\cdot\vec b}_{|\vec b|^2}$

$= |\vec a|^2+|\vec b|^2$.

Projeksjon:



$\vec p$ er Projeksjonen av $\vec a$ ned på $\vec b$

$\vec p$ er en vektor parallell med $\vec b$ slik at $\vec a-\vec p \perp \vec b$.

Hvis $\vec a$ og $\vec b$ er to n-tupler, så er projeksjonen $\vec p$ det n-tuppel parallell med $\vec b$ slik at $\vec a-\vec p \perp \vec b$

$\vec p$ parallell med $\vec b$: $\vec p = t\vec b$

I tillegg er $\vec a-\vec p \perp \vec b$, dvs $(\vec a-\vec p)\cdot\vec b = 0$. Dette gir

$$(\vec a - t\vec b)\cdot\vec b = 0.$$

$$\vec a\cdot\vec b - t\,\vec b\cdot\vec b = 0 \Rightarrow t = \frac{\vec a\cdot\vec b}{\vec b\cdot\vec b} = \frac{\vec a\cdot\vec b}{|\vec b|^2}$$

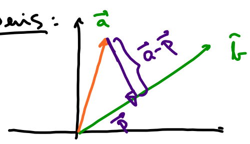
Altså er $\vec p = \frac{\vec a\cdot\vec b}{|\vec b|^2}\cdot\vec b$ Projeksjonen av $\vec a$ ned $\vec b$.

Lengden til $|\vec p| = \left|\underbrace{\frac{\vec a\cdot\vec b}{|\vec b|^2}}_{t}\vec b\right| = \frac{|\vec a\cdot\vec b|}{|\vec b|^{\cancel{2}}}\cancel{|\vec b|} = \frac{|\vec a\cdot\vec b|}{|\vec b|}$

Schwarz' ulikhet: For to n-tupler $\vec a, \vec b$ gjelder

$$|\vec a\cdot\vec b| \le |\vec a||\vec b|$$

Bevis:



Pytagoras: $|\vec a|^2 = |\vec p|^2 + \underbrace{|\vec a-\vec p|^2}_{\ge 0}$

$\ge |\vec p|^2 = \frac{|\vec a\cdot\vec b|^2}{|\vec b|^2}$

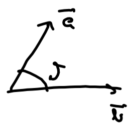
dvs $|\vec a|^2 \ge \frac{|\vec a\cdot\vec b|^2}{|\vec b|^2} \quad |\vec b|^2$

$$|\vec a|^2|\vec b|^2 \ge |\vec a\cdot\vec b|^2$$

dvs $|\vec a||\vec b| \ge |\vec a\cdot\vec b|$

Kuriositet: Husk fra R1/R2: $\vec a\cdot\vec b = |\vec a||\vec b|\cos\theta$

$\cos\theta = \frac{\vec a\cdot\vec b}{|\vec a||\vec b|}$

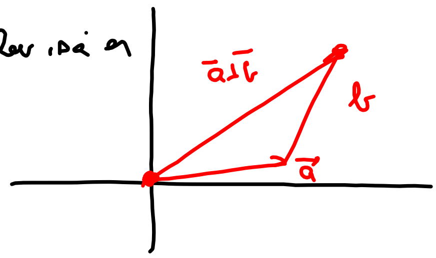


Vi definerer vinkelen θ mellom to n-tupler $\vec a$ og $\vec b$ ved

$$\cos\theta = \underbrace{\frac{\vec a\cdot\vec b}{|\vec a||\vec b|}}_{\text{mellom } -1 \text{ og } 1 \text{ ifølge.}} \qquad 0 \le \theta \le \pi.$$

<u>Trekantulikheten:</u> Hvis $\vec{a}, \vec{b}$ er to n-tupler, så er

$$|\vec{a}+\vec{b}| \leq |\vec{a}| + |\vec{b}|$$



$$|\vec{a}+\vec{b}| \leq |\vec{a}| + |\vec{b}|$$

<u>Bevis:</u>

$$|\vec{a}+\vec{b}|^2 = (\vec{a}+\vec{b}) \cdot (\vec{a}+\vec{b})$$

$$= \vec{a}\cdot\vec{a} + \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{a} + \vec{b}\cdot\vec{b}$$

$$\leq |\vec{a}|^2 + |\vec{a}||\vec{b}| + |\vec{a}|\cdot|\vec{b}| + |\vec{b}|^2$$

$$= |\vec{a}|^2 + 2|\vec{a}||\vec{b}| + |\vec{b}|^2 = (|a| + |b|)^2$$

Altså: $|\vec{a}+\vec{b}|^2 \leq (|\vec{a}|+|\vec{b}|)^2$ Σ veiens!

$$|\vec{a}+\vec{b}| \leq |\vec{a}| + |\vec{b}|$$