Gruppene begynner!

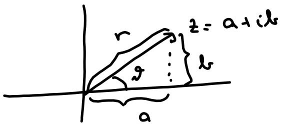
Snublegruppen: Ma 16^{15} - 18 i Aud5VB

Grublegruppen: On 16^{15} - 18 i NHA 1119

Seksjon 3.3

Liten oppsummering:

$$z = \underbrace{a + ib}_{\text{kartesisk form}} = \underbrace{r\cos\theta + ir\sin\theta}_{\text{polarform}} = \underbrace{re^{i\theta}}_{\text{eksponentiell form}}$$



De Moivres formel:

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

$$(e^{i\theta})^n = e^{in\theta}$$

Eksempel: $\cos 3\theta + i\sin 3\theta = (\cos\theta + i\sin\theta)^3$

$= \cos^3\theta + 3\cos^2\theta\, i\sin\theta + 3\cos\theta\, i^2\sin^2\theta + i^3\sin^3\theta$

$= \cos^3\theta + i\,3\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$

$= (\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\cos^2\theta\sin\theta - \sin^3\theta)$

$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$

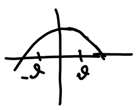
$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$

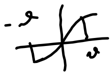
Komplekse trigonometriske funksjoner

Hva skal vi mene med $\cos z$ og $\sin z$ når z er komplekse tall.

Husk: $e^{i\theta} = \cos\theta + i\sin\theta$

$e^{-i\theta} = \cancel{\cos(-\theta) + i\sin(-\theta)} = \cos\theta - i\sin\theta$

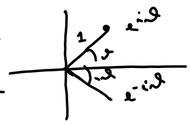




Legger sammen:

$e^{i\theta} + e^{-i\theta} = 2\cos\theta \Rightarrow \cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$

& Subtraherer: $e^{i\theta} - e^{-i\theta} = 2i\sin\theta \Rightarrow \sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$



$e^{-i\theta} = \overline{e^{i\theta}}$

$y'' + ay' + by = 0$

$y = Ce^{r_1 x} + De^{r_2 x}$

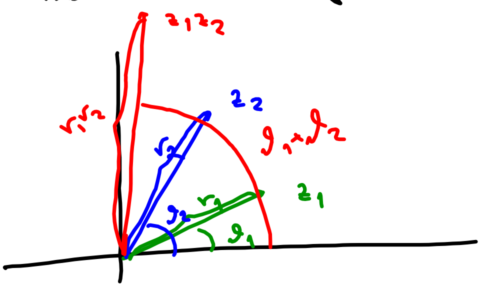
$y = Ce^{rx}\cos dx + De^{rx}\sin dx$

Vi definerer $\cos z$ og $\sin z$ for komplekse z ved

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

Seksjon 3.4

Geometrisk tolkning av multiplikasjon



$z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$

$z_1 z_2 = r_1 r_2 e^{i\theta_1} \cdot e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$

Divisjon: $\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i\theta_1} e^{-i\theta_2}$

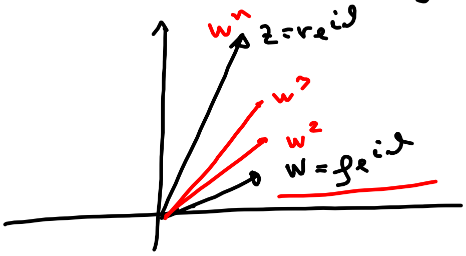
$= \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$

Potenser: $z = re^{i\theta}$, $z^n = \underbrace{z \cdot z \cdots z}_{n} = r^n e^{in\theta}$

$z^n = re^{i\theta} \cdot re^{i\theta} \cdots re^{i\theta} = r^n (e^{i\theta})^n = r^n e^{in\theta}$.

Definisjon: Anta at z er et komplekst tall. Vi kaller w en n-te rot av z dersom $w^n = z$.

Hva betyr dette? $z = re^{i\theta}$, $w = \rho e^{i\varphi}$



Vi trenger $w^n = z$, dvs

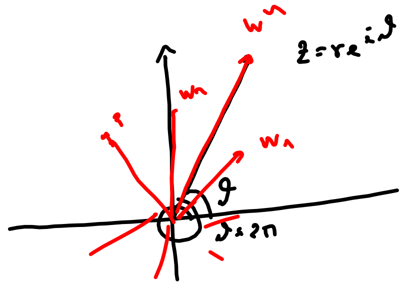
$\underbrace{\rho^n e^{in\varphi}}_{w^n} = re^{i\theta}$

OK hvis $\rho^n = r$ og $n\varphi = \theta$,

dvs $\rho = r^{1/n}$ og $\varphi = \frac{\theta}{n}$.

Altså er $w_0 = r^{1/n} e^{i\frac{\theta}{n}}$ er en n-te rot av z.

Finnes det flere n-te røtter?



Hva hvis w^n treffer z i andre omløp:

$w^n = z$

$\rho^n e^{in\varphi} = re^{i(\theta+2\pi)}$

OK $\rho^n = r$ og $n\varphi = \theta + 2\pi$

$\rho = r^{1/n}$ $\varphi = \frac{\theta+2\pi}{n}$

Ny rot: $w_1 = r^{1/n} e^{i\frac{\theta+2\pi}{n}}$

Hva med treff i neste omløp: $w_2 = r^{1/n}$, $\varphi = \frac{\theta+4\pi}{n}$

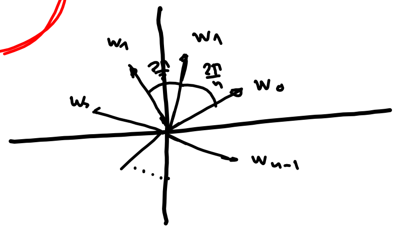
Oppsummering: $z = re^{i\theta}$

n-te røtter:

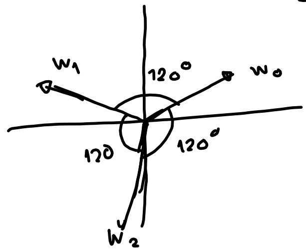
$$w_0 = r^{1/n} e^{i\frac{\theta}{n}}$$
$$w_1 = r^{1/n} e^{i\frac{\theta+2\pi}{n}}$$
$$w_2 = r^{1/n} e^{i\frac{\theta+4\pi}{n}}$$
$$\vdots$$
$$w_k = r^{1/n} e^{i\frac{\theta+2k\pi}{n}}$$
$$\vdots$$
$$w_n = r^{1/n} e^{i\frac{\theta+2n\pi}{n}}$$

2π

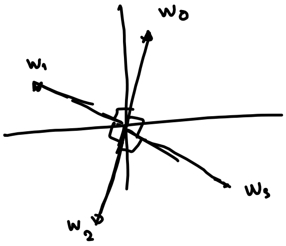
z har n forskjellige n-te røtter $w_0, w_1, \ldots, w_{n-1}$



Tredjerøtter:

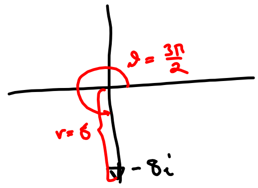


Fjerderøtter:



Eksempel: Finn tredjerøttene $z = -8i$

Skriver z på eksponentialform:



$z = 8e^{i\frac{3\pi}{2}}$

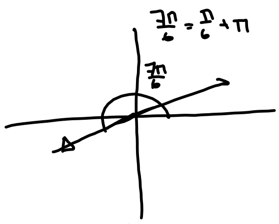
Vi får $\rho = \sqrt[3]{8} = 2$, $\varphi = \frac{\frac{3\pi}{2}}{3} = \frac{\pi}{2}$

Første rot: $w_0 = \rho e^{i\varphi} = 2e^{i\frac{\pi}{2}} = 2(\underbrace{\cos\tfrac{\pi}{2}}_{0} + \underbrace{i\sin\tfrac{\pi}{2}}_{1}) = \underline{\underline{2i}}$

Andre rot: $w_1 = 2e^{i(\frac{\pi}{2}+\frac{2\pi}{3})} = 2e^{i(\frac{3\pi}{6}+\frac{4\pi}{6})} = 2e^{i\frac{7\pi}{6}}$

$= 2(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6})$

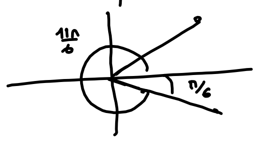
$= 2(-\frac{\sqrt{3}}{2} - i\frac{1}{2}) = \underline{\underline{-\sqrt{3} - i}}$

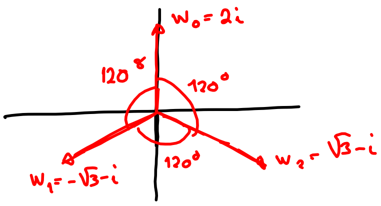


Tredje rot: $w_2 = 2e^{i(\frac{\pi}{2}+\frac{4\pi}{3})} = 2e^{i(\frac{3\pi}{6}+\frac{8\pi}{3})}$

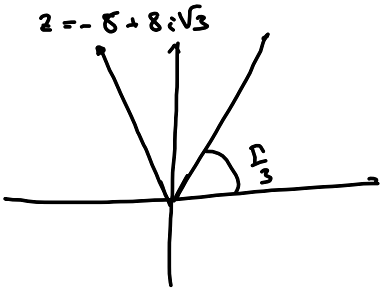
$= 2e^{i\frac{11\pi}{6}} = 2(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6})$

$= 2(\frac{\sqrt{3}}{2} - i\frac{1}{2}) = \sqrt{3} - i.$





Eksempel: Finn fjerderøttene til $z = -8 + 8i\sqrt{3}$



Finner polarkoordinatene:

$$r = \sqrt{(-8)^2 + (8\sqrt{3})^2} = \sqrt{8^2 \underbrace{(1+(\sqrt{3})^2)}_{4}}$$

$$= \sqrt{8^2 \cdot 4} = \sqrt{8^2} \cdot \sqrt{4} = 8 \cdot 2 = \underline{\underline{16}}$$

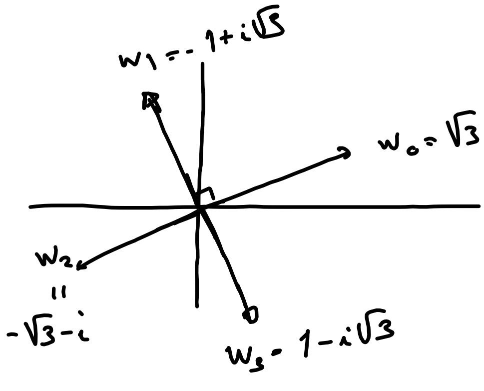
$$\sin\theta = \frac{b}{r} = \frac{8\sqrt{3}}{16} = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{2\pi}{3}$$ siden z er i andre kvadrant.

Finner så $\rho = r^{1/4} = 16^{1/4} = \underline{2}$

$$\varphi = \frac{\theta}{4} = \frac{\frac{2\pi}{3} \cdot 3}{4 \cdot 3} = \frac{2\pi}{12} = \frac{\pi}{6}$$

Finner røttene: $w_0 = \rho e^{i\varphi} = 2e^{i\frac{\pi}{6}} = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

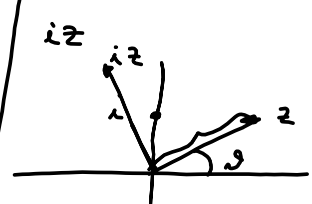
$$= 2\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) = \underline{\underline{\sqrt{3} + i}}$$



$$w_1 = i \cdot w_0 = i(\sqrt{3} + i)$$

$$= i\sqrt{3} - 1$$

$$= \underline{\underline{-1 + i\sqrt{3}}}$$



iz har lengde $|z|$ og argument $(\theta + \frac{\pi}{2})$

Komplekse annengradsligninger

Vanlig annengradsligning: $ax^2+bx+c=0 \qquad a,b,c \in \mathbb{R}$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

Komplekse annengradsligning: $az^2+bz+c=0 \qquad a,b,c \in \mathbb{C}$

$$z = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \quad ?$$

Utledning:

$$az^2+bz+c = 0 \mid 4a$$

$$\underbrace{4a^2z^2+4abz}+4ac = 0$$

$$\underbrace{(2az+b)^2}_{4a^2z^2+4abz+b^2}-b^2+4ac = 0$$

$$(2az+b)^2 = b^2-4ac$$

$$2az+b = \pm\sqrt{b^2-4ac}$$

$$2az = -b \pm \sqrt{b^2-4ac}$$

$$z = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$