Eksamen 2017, oppgave 1

$$f(x,y) = x e^{xy}$$

a) $$\frac{\partial f}{\partial x} = 1 \cdot e^{xy} + x e^{xy} \cdot y = e^{xy}(1+xy)$$

$$\frac{\partial f}{\partial y} = x e^{xy} \cdot x = x^2 e^{xy}$$

b) Regner ut gradienten $\nabla f(x,y) = (e^{xy}(1+xy), x^2 e^{xy})$
$= e^{xy}(1+xy, x^2)$

$$\nabla f(2,1) = e^{2\cdot 1}(1+2\cdot 1, 2^2) = e^2(3,4)$$

Funksjonen vokser raskest i retning $(3,4)$ evt $e^2(3,4)$.

Stigningstallet i retning $\vec{u} = \frac{\nabla f(2,1)}{|\nabla f(2,1)|}$ er

$$|\nabla f(2,1)| = e^2 \underbrace{\sqrt{3^2+4^2}}_{5} = 5e^2$$

Hvorfor:

$$f'((2,1);\vec{u}) = \nabla f(2,1) \cdot \frac{\nabla f(2,1)}{|\nabla f(2,1)|} = \frac{|\nabla f(2,1)|^2}{|\nabla f(2,1)|} = |\nabla f(2,1)|$$

Eksamen 2010, oppgave 13

$$f(x) = \tfrac{1}{2} x|x| = \begin{cases} \tfrac{1}{2}x^2 & \text{hvis } x \geq 0 \\ -\tfrac{1}{2}x^2 & \text{hvis } x \leq 0 \end{cases}$$

For $x > 0$, er $f'(x) = x$

For $x < 0$, er $f'(x) = -x$

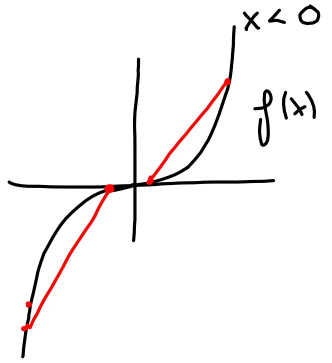
Hva skjer i $x = 0$:

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{\frac{1}{2}\cancel{x}|x|}{\cancel{x}} = 0.$$

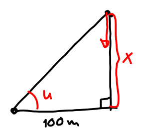
f er deriverbar på hele $\mathbb{R}$!

Konveks/konkav: $x > 0$, så $f''(x) = (x)' = 1 > 0$ konveks

$x < 0$, så $f''(x) = (-x)' = -1 < 0$ konkav



2017, oppgave 3



$u=\frac{\pi}{4}$, da endrer u seg med en fart på 0.03 rad/sek.

$$\tan u = \frac{x}{100}$$

Deriverer mhp t:

$$\frac{1}{\cos^2 u}\, u' = \frac{1}{100}\, x'$$

$$x' = \frac{100}{\cos^2 u}\, u' = \frac{100}{\cos^2 \frac{\pi}{4}} (-0.03) = 200(-0.03) = \underline{-6 \text{ m/s}}$$

$$\left(\frac{\sqrt{2}}{2}\right)^2 = \frac{1}{2}$$

Fart: 6 m/s

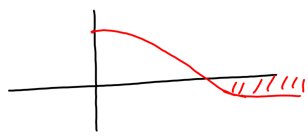
2020, oppgave 6: $f: [0,\infty) \to \mathbb{R}$, kont., avtagende og

$$\int_0^\infty f(x)\,dx \text{ konvergerer.}$$

Vis at $\int_0^\infty f(x)^2\,dx$ konvergerer

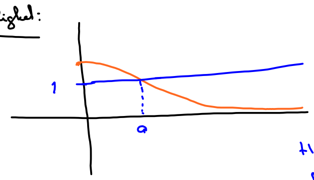
Muligheter:

$\lim f(x) > 0 \Rightarrow \int^\infty f(x)\,dx = \infty$



$\lim f(x) < 0 \Rightarrow \int_0^\infty f(x)\,dx = -\infty$

Eneste mulighet:



$\lim_{x\to\infty} f(x) = 0$

Hvis $f(x) < 1$ så $\underline{f(x)^2 < f(x)}$

$$\int_0^\infty f(x)^2\,dx = \int_0^a f(x)^2\,dx + \underline{\int_a^\infty f(x)^2\,dx}$$

$$\le \underbrace{\int_0^a f(x)^2\,dx} + \underbrace{\int_a^\infty f(x)\,dx} < \infty. \quad \text{Altså konvergens.}$$

2018, oppgave 6: $\vec{F}: \mathbb{R}^2 \to \mathbb{R}$ gitt ved

$$\vec{F}(x,y) = \begin{pmatrix} e^{x^2+y} - 1 \\ x^2+y \end{pmatrix} = \begin{pmatrix} F_1(x,y) \\ F_2(x,y) \end{pmatrix}$$

Jacobimatrisen:

$$\vec{F}'(x,y) = \begin{pmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{pmatrix} = \begin{pmatrix} e^{x^2+y}\cdot 2x & e^{x^2+y}\cdot 1 \\ 2x & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2x e^{x^2+y} & e^{x^2+y} \\ 2x & 1 \end{pmatrix}$$

Begrunn at det finnes en M slik at

$$\lim_{(x,y)\to(0,0)} \frac{\vec{F}(x,y) - M\binom{x}{y}}{\sqrt{x^2+y^2}} = 0$$

$\vec{F}(\vec{0}) = \vec{0}$

$\vec{r} = \binom{x}{y}$: $\displaystyle \lim_{\vec{r}\to\vec{0}} \frac{\vec{F}(\vec{r}) - M\vec{r}}{|\vec{r}|}$

$$= \lim_{\vec{r}\to\vec{0}} \frac{\vec{F}(\vec{0}+\vec{r}) - \vec{F}(\vec{0}) - M\vec{r}}{|\vec{r}|}$$

Sett $M = \vec{F}'(\vec{0})$

$$= \lim_{\vec{r}\to\vec{0}} \frac{\overbrace{\vec{F}(\vec{0}+\vec{r}) - \vec{F}(\vec{0}) - \vec{F}'(\vec{0})\vec{r}}^{\vec{\sigma}(\vec{r})}}{|\vec{r}|}$$

$\longrightarrow 0$ fordi $\vec{F}$ er deriverbar.

Siden alle partiellderiverte $\frac{\partial F_i}{\partial x_j}$ er kontinuerlige er $\vec{F}$ deriverbar.

2017, oppgave 6

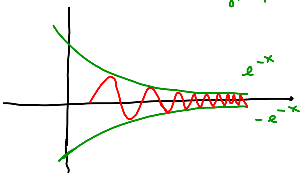
a) $g(x) = e^{-x}\sin(e^x)$

$g'(x) = -e^{-x}\sin(e^x) + \cancel{e^{-x}}\cos(e^x)\cdot\cancel{e^x}$

$= -e^{-x}\sin(e^x) + \cos(e^x)$.

Dermed: $\lim_{x\to\infty} g(x) = \lim_{x\to\infty}(\underbrace{e^{-x}}_{\to 0}\underbrace{\sin(e^x)}_{\text{begrenset}}) = 0$

$\lim_{x\to\infty} g'(x) = \lim_{x\to\infty}[\underbrace{-e^{-x}\sin(e^x)}_{\to 0} + \underbrace{\cos(e^x)}_{\text{svinger mellom ytterpunkter -1 og 1}}]$ grensen finnes ikke



$g(x) = e^{-x}\sin(e^x)$

$\boxed{f''(x)(b-x)}$

b) $(f(x) + f'(x)(b-x))'$

$= \cancel{f'(x)} + f''(x)(b-x) + \cancel{f'(x)(-1)} = f''(x)(b-x)$

Dermed

$\int_a^b f''(x)(b-x)\,dx = [f(x) + f'(x)(b-x)]_a^b$

$f(b) + \cancel{f'(b)(b-b)} - f(a) - f'(a)(b-a)$

$= f(b) - f(a) - f'(a)(b-a)$ $\quad \sigma(\vec r) = f(\vec a + \vec r) - f(\vec a) - f'(\vec a)\vec r$

c) $|f''(t)| \le M$ for alle t og vis

$\frac{1}{2}M(b-a)^2 \ge |f(b) - f(a) - f'(a)(b-a)|$

Vi har

$|f(b)-f(a)-f'(a)(b-a)| = |\int_a^b f''(x)(b-x)\,dx|$

$\le \int_a^b M(b-x)\,dx = [-\frac{1}{2}M(b-x)^2]_a^b = -\frac{1}{2}M(b-b)^2 + \frac{1}{2}M(b-a)^2 = \frac{1}{2}M(b-a)^2$

Ser at

$\frac{1}{2}M(b-a)^2 \ge |\overbrace{(f(b)-f(a))}^{\text{avstand mellom}} - f'(a)(b-a)|$

$\ge |\underbrace{|f(b)-f(a)| - |f'(a)|(b-a)}_{\text{avstanden mellom.}}|$

Dermed

$|f(b)-f(a)| \ge |f'(a)|(b-a) - \frac{1}{2}M(b-a)^2$

$= (|f'(a)| - \frac{1}{2}M(b-a))(b-a)$

$|f'(a)|(b-a)$ $\quad |f'(a)|(b-a) - \frac{1}{2}M(b-a)^2$

$\frac{1}{2}M(b-a)^2$ $\quad |f(b)-f(a)|$

d) $|f'(a)| \ge \varepsilon$ og $b = a + \frac{\varepsilon}{M}$

Vis at $|f(b)-f(a)| \ge \frac{\varepsilon^2}{2M}$:

$|f(b)-f(a)| \ge |\underbrace{|f'(a)|}_{\ge\varepsilon} - \frac{1}{2}M\underbrace{(b-a)}_{\varepsilon/M}|\underbrace{(b-a)}_{\varepsilon/M}$

$\ge |\varepsilon - \frac{1}{2}\cancel{M}\frac{\varepsilon}{\cancel{M}}|\frac{\varepsilon}{M} = \frac{\varepsilon}{2}\cdot\frac{\varepsilon}{M} = \frac{\varepsilon^2}{2M}$ ←

Anta for motsigelse at $\lim_{x\to\infty} f'(x) \neq 0$. Da finnes det en $\varepsilon > 0$ slik at $|f'(x)| \ge \varepsilon$ for vilkårlig store x. Dermed finnes det en voksende følge $\{a_n\}$ slik at $a_n \to \infty$ og $|f'(a_n)| \ge \varepsilon$. Sett $b_n = a_n + \frac{\varepsilon}{M}$. Da a

2011, oppgave 8: $\int_0^\infty \frac{\arctan x}{x+1}\,dx$ $\qquad \frac{\arctan x}{x+1} \approx \frac{\frac{\pi}{2}}{x}$

Grensesammenligningstesten: $g(x) = \frac{1}{x}$ ($\int \frac{1}{x}dx$ divergerer).

$$\lim_{x\to\infty} \frac{\frac{\arctan x}{x+1}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{x\arctan x}{x+1} = \lim_{x\to\infty} \underbrace{\arctan x}_{\frac{\pi}{2}} \lim_{x\to\infty} \underbrace{\frac{x}{x+1}}_{1} = \frac{\pi}{2} > 0$$

Integralet divergerer.

2011, oppgave 9: $\int_0^\infty \frac{1}{x^2+2x+2}\,dx$ $\qquad \frac{1}{x^2+2x+2} \sim \frac{1}{x^2}$

$$= \lim_{a\to\infty} \int_0^a \frac{1}{(x^2+2x+1)+1}\,dx = \lim_{a\to\infty} \int_0^a \frac{1}{(x+1)^2+1}\,dx$$

$u = x+1$
$du = dx$
$u(0) = 1$
$u(a) = a+1$

$$= \lim_{a\to\infty} \int_1^{a+1} \frac{1}{u^2+1}\,du = \lim_{a\to\infty} \left[\arctan u\right]_1^{a+1}$$

$$= \lim_{a\to\infty} \left[\underbrace{\arctan(a+1)}_{\frac{\pi}{2}} - \underbrace{\arctan(1)}_{\frac{\pi}{4}}\right] = \frac{\pi}{2} - \frac{\pi}{4} = \underline{\underline{\frac{\pi}{4}}}$$

Kont 2014, nr 13: a) $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$

$$AB = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$$

$\det(AB) = \det(A)\det(B)$

$\det(AB) = (ae+bg)(cf+dh) - (af+bh)(ce+dg) =$

$\det(A)\det(B) = (ad-bc)(eh-fg) =$

b) Anta C er et kvadrat: $C = D^2$

$\det C = \det D^2 = \det D \det D = (\det D)^2 \geq 0$.

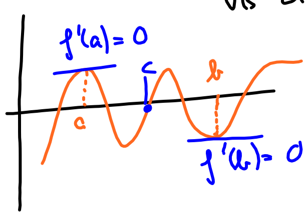
Vis at $C = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ ikke er et kvadrat.

$\det C = 1\cdot 1 - 2\cdot 2 = -3 < 0$ (ikke et kvadrat).

Kont 2014, oppgave 14: $a < b$, lokale ekstrempunkt.

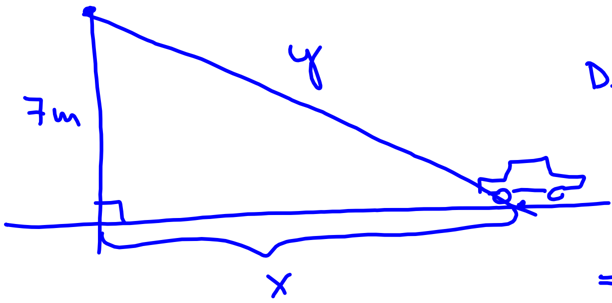
Vis at det finnes c, $a < c < b$

der $f''(c) = 0$



Ifølge Rolles teorem brukt på f', finnes det en $c \in (a,b)$ der $f''(c) = 0$.

Kont 2003, del 2, oppgave 4



Pytagoras:

$y^2 = x^2 + 7^2 = x^2 + 49$

Deriverer

$\cancel{2}yy' = \cancel{2}xx'$

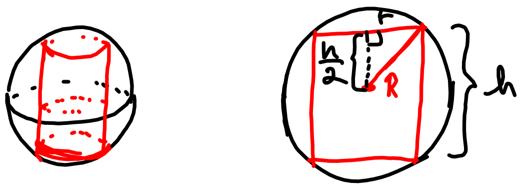
$x' = \frac{yy'}{x} = \frac{y \cdot (-30 \, m/s)}{24}$

$= \frac{25(-30)}{24} = -31.25 \, m/s$

$\boxed{x = 24 \, m \quad y' = -30 \, m/s}$

Når $x = 24$, då er $y^2 = 24^2 + 7^2 = 576 + 49 = 625 = 25^2 \Rightarrow y = 25$

Prøveeksamen 2, 2017, oppgave 7



$V = \pi r^2 h$

Pytagoras: $r^2 + \left(\frac{h}{2}\right)^2 = R^2$

$\Rightarrow r^2 = R^2 - \frac{h^2}{4}$

$V(h) = \pi \left(R^2 - \frac{h^2}{4}\right)h$

$= \pi \left(R^2 h - \frac{h^3}{4}\right)$

Deriverer: $V'(h) = \pi \left(R^2 - \frac{3h^2}{4}\right)$

$0 = V'(h) \Rightarrow R^2 = \frac{3h^2}{4} \Rightarrow h^2 = \frac{4}{3}R^2 \Rightarrow h = \frac{2}{\sqrt{3}}R$

$V_{max} = \pi \left(R^2 \frac{2}{\sqrt{3}} R - \frac{\frac{2\cancel{8}}{3\sqrt{3}}R^3}{\cancel{4}}\right) = \pi R^3 \left(\frac{2}{\sqrt{3}} - \frac{2}{3\sqrt{3}}\right)$

$= \frac{2}{\sqrt{3}} \pi R^3 \underbrace{\left(1 - \frac{1}{3}\right)}_{\frac{2}{3}} = \frac{4}{3\sqrt{3}} \pi R^3$