Komplekse annengradsligninger

$az^2 + bz + c = 0$, $a, b, c \in \mathbb{C}$

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ – de to kvadratrøttene til $b^2 - 4ac$

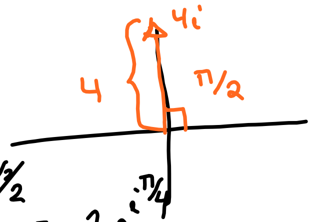
Eks: $z^2 + 2iz + (-1-i) = 0$

$a = 1$
$b = 2i$
$c = -1-i$

$$z = \frac{-2i \pm \sqrt{(2i)^2 - 4 \cdot 1(-1-i)}}{2 \cdot 1}$$

$$= \frac{-2i \pm \sqrt{-4 + 4 + 4i}}{2} = \frac{-2i \pm \sqrt{4i}}{2}$$

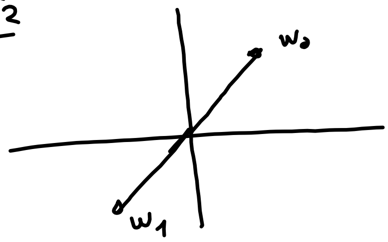
Må finne kvadratrøtten til $4i$: $r = 4$, $\theta = \frac{\pi}{2}$



Finner den første kvadratroten: $w_0 = 4^{1/2} e^{i\cdot\pi/2} = 2e^{i\pi/4}$

$$= 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = 2\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = \sqrt{2} + i\sqrt{2}$$

$$W_1 = -w_0 = -\sqrt{2} - i\sqrt{2}$$



$$z = \frac{-2i \pm (\sqrt{2} + i\sqrt{2})}{2}$$

$$= \begin{cases} \dfrac{-2i + \sqrt{2} + i\sqrt{2}}{2} = \dfrac{\sqrt{2}}{2} + i\dfrac{(\sqrt{2}-2)}{2} \\[2ex] \dfrac{-2i - (\sqrt{2} + i\sqrt{2})}{2} = -\dfrac{\sqrt{2}}{2} - i\dfrac{\sqrt{2}+2}{2} \end{cases}$$

Oppgaver

3.1.6

$$z + w = 2i \quad (*)$$
$$z - w = 3 + i \quad (**)$$

Addisjon: $2z = 2i + 3 + i = 3 + 3i$

$$z = \frac{3+3i}{2} = \frac{3}{2} + \frac{3}{2}i$$

Finner w fra $(*)$: $w = 2i - z = 2i - \frac{3}{2} - \frac{3}{2}i = -\frac{3}{2} + \frac{i}{2}$

3.1.9: Vis at $\bar{z}w$ og $z\bar{w}$ er konjugerte.

$$\overline{\bar{z}w} = \bar{\bar{z}}\,\bar{w} = z \cdot \bar{w}.$$

3.2.11a) Skisser området

$$A = \{z : 2\,\mathrm{Re}(z) < |z|^2\}$$

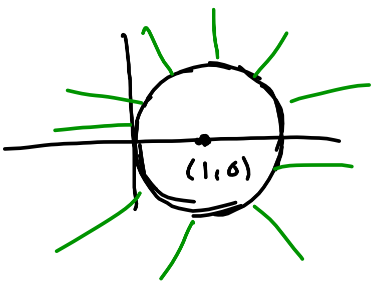
$z = a + ib$
$\mathrm{Re}(z) = a$ realdel
$\mathrm{Im}(z) = b$ imaginærdel.

La $z = x + iy$: $2\,\mathrm{Re}(z) = 2x$
$|z|^2 = \left(\sqrt{x^2+y^2}\right)^2 = x^2 + y^2$

Området A består dermed av de punktene $z = x + iy$ slik at

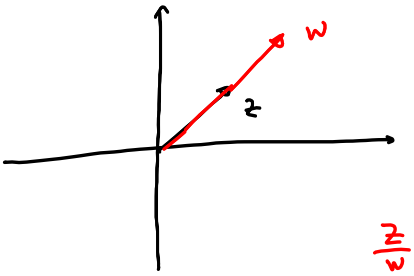
$$2x < x^2 + y^2$$
$$0 < x^2 - 2x + y^2$$
$$0 < \underbrace{x^2 - 2x + 1}_{(x-1)^2} - 1 + y^2$$
$$1 < (x-1)^2 + y^2$$

$$\underbrace{(x-1)^2 + y^2 = 1^2}_{\text{sirkel om } (1,0) \text{ med radius } 1}$$



Nesten 3.2.12 : Hva betyr det at $\frac{z}{w}$ er reell?

—— " —— $\frac{z}{w}$ er imaginær?



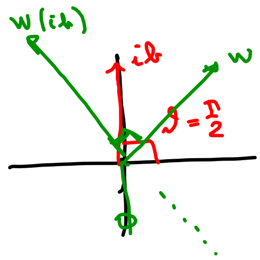
z og w på linje, betyr

$z = tw \Rightarrow \frac{z}{w} = t$ reell

$t \in \mathbb{R}$

$\frac{z}{w}$ er reell $\Leftrightarrow$ z og w er parallelle.

Hva skjer når vi ganger et komplekst tall w med et et imaginært tall ib? Jo, svaret står rett vinkel på w.

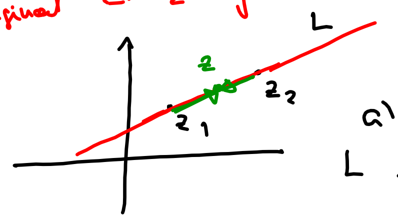


Hva betyr det at $\frac{z}{w} = ib$?

$z = (ib)w$

$\frac{z}{w}$ imaginær $\Leftrightarrow$ z og w står 90° på hverandre.

3.2.17 a)

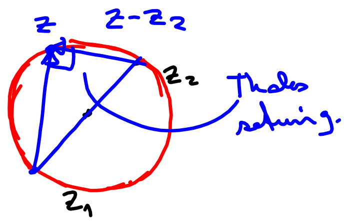


$z_1 \neq z_2$

a) Vis at z ligger på linjen L hvis og bare hvis $\frac{z-z_1}{z-z_2}$ er reell eller $z = z_2$

z ligger på linjen hvis $z - z_1$ og $z - z_2$ er parallelle, dvs hvis $\frac{z-z_1}{z-z_2}$ er et reell tall (eller $z = z_2$)

b) Vis z ligger på sirkelen gjennom z_1 og z med sentrum i $\frac{z_1+z_2}{2}$ hvis og bare hvis $\frac{z-z_1}{z-z_2}$ er imaginær.



z ligger på sirkelperiferien

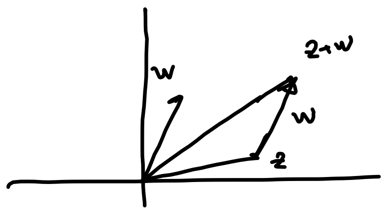
$\Updownarrow$

vinkelen mellom $z - z_1$ og $z - z_2$ er rett.

$\Updownarrow$

$\frac{z-z_1}{z-z_2}$ er imaginær.

3.2.22 Trekantulikheten $|z+w| \le |z|+|w|$



$$\underline{|z+w|^2} = (z+w)\overline{(z+w)} = (z+w)(\bar z + \bar w)$$

$$= \underbrace{z\bar z}_{|z|^2} + \underbrace{z\bar w + w\bar z}_{\text{konjugerte}} + \underbrace{w\bar w}_{|w|^2}$$

$$= |z|^2 + 2\,\mathrm{Re}(z\bar w) + |w|^2$$

$$\le |z|^2 + 2|z\bar w| + |w|^2$$

Fordi lengden til et komplekst tall alltid er større enn eller lik realdelen.

$|z|^2 = z\cdot\bar z$

$a+ib$
$+$
$a-ib$

$$= |z|^2 + 2|z|\underbrace{|\bar w|}_{=|w|} + |w|^2$$

$$= |z|^2 + 2|z||w| + |w|^2 = \underline{(|z|+|w|)^2}$$

dvs $|z+w|^2 \le (|z|+|w|)^2$ som gir

$$|z+w| \le |z|+|w|.$$

3.3.11 a) $2\sin\frac{z+w}{2}\cos\frac{z-w}{2} = \sin z + \sin w$

$$\sin u = \frac{e^{iu}-e^{-iu}}{2i} \qquad \cos u = \frac{e^{iu}+e^{-iu}}{2}$$

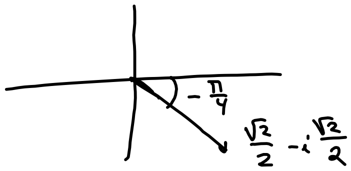
$$2\sin\frac{z+w}{2}\cos\frac{z-w}{2} = \cancel{2}\,\frac{e^{i\frac{z+w}{2}} - e^{-i\frac{z+w}{2}}}{2i}\cdot\frac{e^{i\frac{z-w}{2}} + e^{-i\frac{z-w}{2}}}{\cancel{2}}$$

3.4.6 $\quad z^3 = \frac{\sqrt{2}}{1+i}$ løs

Omskrivning: $\frac{\sqrt{2}}{1+i} = \frac{\sqrt{2}\,(1-i)}{(1+i)(1-i)} = \frac{\sqrt{2}\,(1-i)}{\underbrace{1^2-i^2}_{2}} = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$

For å finne 3. røttene til $\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$, skriver vi tallet på polarform.

$\theta = -\frac{\pi}{4}$, $r = \sqrt{\left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2} = \sqrt{\frac{1}{2}+\frac{1}{2}} = \underline{\underline{1}}$



Hva blir tredjerøttene:

$w_0 = 1^{1/3} e^{i(-\frac{\pi}{12})} = e^{-i\frac{\pi}{12}} = \cos(-\frac{\pi}{12}) + i\sin(-\frac{\pi}{2})$

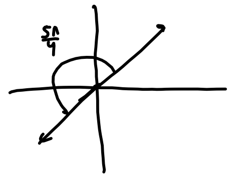
$w_1 = 1 \cdot e^{i(-\frac{\pi}{12}+\frac{2\pi}{3})} = e^{i(-\frac{\pi}{12}+\frac{8\pi}{12})} = e^{i(\frac{7\pi}{12})}$

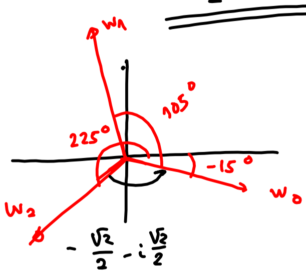
$= \cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}$.

$w_2 = 1 \cdot e^{i(-\frac{\pi}{12}+\frac{4\pi}{3})} = e^{i(-\frac{\pi}{12}+\frac{16\pi}{12})} = e^{i\frac{15}{12}\pi} = e^{i\frac{5}{4}\pi}$

$= \cos(\frac{5\pi}{4}) + i\sin(\frac{5\pi}{4})$ $\qquad \frac{5\pi}{4} = \frac{\pi}{4} + \pi$

$= \underline{\underline{-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}}}$





$w_0 = w_2 e^{i\frac{2\pi}{3}}$

$= \left(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

$= \left(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$

$= \frac{\sqrt{2}}{4} - i\frac{\sqrt{6}}{4} + i\frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4}$

$= \frac{\sqrt{6}+\sqrt{2}}{4} + i\left(\frac{\sqrt{2}-\sqrt{6}}{4}\right)$

$w_1 = w_0 e^{i\frac{2\pi}{3}}$

Oblig 1.