Løsningsforslag eksamen Mat1100 tirs 11/12 2012

Oppgave 1

$$\begin{array}{cc|cc} & & 3 & 4 \\ & & 2 & -1 \\ \hline 3 & 4 & 17 & 8 \\ 2 & -1 & 4 & 9 \end{array}$$

$\boxed{C}$

Oppgave 2

$$f(x,y) = x^5y^7 - 4x^2 + y$$

$$\frac{\partial f}{\partial x} = 5x^4y^7 - 8x$$

$\boxed{B}$

Oppgave 3

$$\begin{vmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ -1 & 2 & 3 \end{vmatrix} = 2 \cdot \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} - 3 \cdot \begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix} + (-1) \cdot \begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix}$$

$$= 2 \cdot (3-2) - 3 \cdot (3+1) - 1(2+1)$$

$$= 2 - 12 - 3 = -13$$

$\boxed{A}$

Oppgave 4

$$\nabla f = \left(e^{2xy} \cdot 2y - \frac{1}{2}e^{xz} \cdot z,\ e^{2xy} \cdot 2x,\ -\frac{1}{2}e^{xz} \cdot x\right)$$

$$\nabla f(0,1,5) = \left(2 - \frac{5}{2},\ 0,\ 0\right)$$

$\boxed{A}$

Oppgave 5

$$f(x,y) = x^2y - xy^2$$

$$\nabla f = (2xy - y^2,\ x^2 - 2xy)$$

$$\nabla f(4,1) = (7,\ 8)$$

$$\nabla f(4,1)\cdot(1,1) = (7,8)\cdot(1,1) = 15 \qquad \boxed{B}$$

Oppgave 6

$$\int_0^1 \frac{(\arctan x)^7}{1+x^2}\,dx = \int_0^{\pi/4} \frac{u^7}{\cancel{1+x^2}}\,(\cancel{1+x^2})\,du = \left[\frac{1}{8}u^8\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{8}\left(\frac{\pi}{4}\right)^8$$

$u = \arctan x \qquad \frac{du}{dx} = \frac{1}{1+x^2}$

$du = \frac{1}{1+x^2}\,dx \qquad dx = (1+x^2)\,du$

$x = 0$ gir $u = 0$

$x = 1$ gir $u = \pi/4$

$$\boxed{E}$$

Oppgave 7

$$\vec{F}(x,y) = (xy^2 + 1,\ xy)$$

$$\vec{F}' = \begin{pmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{pmatrix} = \begin{pmatrix} y^2 & 2xy \\ y & x \end{pmatrix}$$

$$\vec{F}'(3,-1) = \begin{pmatrix} 1 & -6 \\ -1 & 3 \end{pmatrix} \qquad \boxed{B}$$

Oppgave 8

$$V = \pi \int_0^{1/2} \left[f'(x)\right]^2 dx = \pi \int_0^{1/2} \frac{1}{\sqrt{1-x^2}}\,dx$$

$$= \pi \left[\arcsin x\right]_0^{1/2} = \pi\left[\frac{\pi}{6} - 0\right] = \frac{\pi^2}{6} \qquad \boxed{C}$$

Oppgave 9

$$\int_0^{1/a} x \sin(\pi a x)\,dx = \left[-\frac{x}{\pi a}\cos(\pi a x)\right]_0^{1/a} + \frac{1}{\pi a}\int_0^{1/a} \cos(\pi a x)\,dx$$

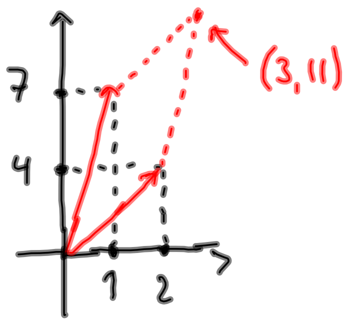
Delvis: $F(x) = x \qquad G'(x) = \sin(\pi a x)$

$F'(x) = 1 \qquad G(x) = -\frac{1}{\pi a}\cos(\pi a x)$

$$= \left[-\frac{1}{\pi a^2}\cos\pi + \frac{1}{\pi a^2}\cos 0\right] + \frac{1}{\pi^2 a^2}\left[\sin(\pi a x)\right]_0^{1/a}$$

$$= \frac{1}{\pi a^2} + \frac{1}{\pi^2 a^2}\left[\sin\pi - \sin 0\right] = \frac{1}{\pi a^2} \qquad \boxed{C}$$

Oppgave 10



$$\begin{vmatrix} 2 & 4 \\ 1 & 7 \end{vmatrix} = 14 - 4 = 10 \qquad \boxed{D}$$

Alternativt: La $\vec{a} = (2,4,0)$

$\vec{b} = (1,7,0)$

og regn ut $|\vec{a} \times \vec{b}|$

Oppgave 11

$$V = 2\pi \int_1^2 x f(x)\,dx = 2\pi \int_1^2 x e^{x^2}\,dx$$

$$= 2\pi \int_1^4 x e^u \frac{1}{2x}\,du$$

$u = x^2 \quad \frac{du}{dx} = 2x$

$du = 2x\,dx \quad dx = \frac{1}{2x}\,du$

$x = 1$ gir $u = 1$
$x = 2$ gir $u = 4$

$$= \pi \cdot \left[e^u\right]_1^4$$

$$= \pi\left(e^4 - e\right)$$

Oppgave 12

La $h(x) = f(x) - g(x)$
$= x + \ln x + 2 - e^x$

Da er $h(1) = 1 + 0 + 2 - e > 0$
$h(10) = 12 + \ln 10 - e^{10} < 22 - 2^{10} < 0$

Siden h er kontinuerlig, finnes ved skjæringssetningen $x \in (1, 10)$ slik at $h(x) = 0$, dvs.

$$f(x) = g(x).$$

Oppgave 13

Vi har

$$\lim_{x\to\infty} \frac{\frac{5x^{3/2}}{\ln x + 4x^{5/2}}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{5x^{5/2}}{\ln x + 4x^{5/2}}$$

$$= \lim_{x\to\infty} \frac{5}{\underbrace{\frac{\ln x}{x^{5/2}}}_{\to 0} + 4} = \frac{5}{4}$$

Siden vi vet at $\int_1^\infty \frac{1}{x}\,dx$ divergerer, divergerer dermed integralet vårt ved grensesammenlikningstesten for integraler.

Alternativt: Kan bruke vanlig sammenlikningstest:

$$\frac{5x^{3/2}}{\ln x + 4x^{5/2}} > \frac{5x^{3/2}}{x^{5/2} + 4x^{5/2}} = \frac{5x^{3/2}}{5x^{5/2}} = \frac{1}{x}$$

Sammenlikner så med $\int_1^\infty \frac{1}{x}\,dx$, som ovenfor.

Oppgave 14

$$\int \frac{\cos x}{\sin^2 x + 6\sin x + 25}\,dx = \int \frac{1}{u^2+6u+25}\,du$$

$$\boxed{\begin{array}{ll} u = \sin x & \frac{du}{dx} = \cos x \\ du = \cos x\, dx & dx = \frac{1}{\cos x}\, du \end{array}}$$

$$= \int \frac{1}{(u^2+6u+9)+16}\,du = \int \frac{1}{(u+3)^2+16}\,du$$

$$= \frac{1}{16}\int \frac{1}{\left[\frac{1}{4}(u+3)\right]^2+1}\,du$$

$$= \frac{1}{16}\int \frac{1}{v^2+1}\cdot 4\,dv = \frac{1}{4}\arctan v + C$$

$$\boxed{\begin{array}{ll} v = \frac{1}{4}(u+3) & \frac{dv}{du} = \frac{1}{4} \\ dv = \frac{1}{4}\,du & du = 4\,dv \end{array}}$$

$$= \frac{1}{4}\arctan\left[\frac{1}{4}(u+3)\right] + C$$

$$= \frac{1}{4}\arctan\left[\frac{1}{4}(\sin x + 3)\right] + C$$

Oppgave 15

a)

$$\begin{cases} x_{n+1} = (\text{nye år } n+1) = (\text{antall som stod år } n) = 0{,}5x_n + 0{,}75y_n \\ y_{n+1} = (\text{gamle år } n+1) = (\text{— " — strøk ---}) = 0{,}5x_n + 0{,}25y_n \end{cases}$$

så $\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 0{,}5 & 0{,}75 \\ 0{,}5 & 0{,}25 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}$ for $n \geq 1$.

b) For $n = 1$ sier formelen

$$M^1 = \frac{1}{5}\begin{pmatrix} 3-\frac{1}{2} & 3+\frac{3}{4} \\ 2+\frac{1}{2} & 2-\frac{3}{4} \end{pmatrix} = \frac{1}{5}\begin{pmatrix} \frac{5}{2} & \frac{15}{4} \\ \frac{5}{2} & \frac{5}{4} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{3}{4} \\ \frac{1}{2} & \frac{1}{4} \end{pmatrix}$$

og dette stemmer. Anta nå at formelen holder for en gitt verdi n. Vi får da

$$M^{n+1} = M^n \cdot M = \frac{1}{5} \begin{array}{cc|cc} & & \frac{1}{2} & \frac{3}{4} \\ & & \frac{1}{2} & \frac{1}{4} \\ \hline 3+2\left(-\frac{1}{4}\right)^n & 3-3\left(-\frac{1}{4}\right)^n & 1 & 2 \\ 2-2\left(-\frac{1}{4}\right)^n & 2+3\left(-\frac{1}{4}\right)^n & 3 & 4 \end{array}$$

der 1 – 4 blir

$$1 = \left[3+2\left(-\tfrac{1}{4}\right)^n\right]\cdot\tfrac{1}{2} + \left[3-3\left(-\tfrac{1}{4}\right)^n\right]\cdot\tfrac{1}{2}$$

$$= \tfrac{3}{2} + \left(-\tfrac{1}{4}\right)^n + \tfrac{3}{2} - \tfrac{3}{2}\left(-\tfrac{1}{4}\right)^n$$

$$= 3 - \tfrac{1}{2}\left(-\tfrac{1}{4}\right)^n$$

$$= 3 + 2\cdot\left(-\tfrac{1}{4}\right)\cdot\left(-\tfrac{1}{4}\right)^n = 3+2\left(-\tfrac{1}{4}\right)^{n+1}$$

$$2 \overset{\text{tilsv.}}{=} 3 - 3\cdot\left(-\tfrac{1}{4}\right)\cdot\left(-\tfrac{1}{4}\right)^n = 3-3\left(-\tfrac{1}{4}\right)^{n+1}$$

$$3 \overset{\text{tilsv.}}{=} 2 - 2\cdot\left(-\tfrac{1}{4}\right)\cdot\left(-\tfrac{1}{4}\right)^n = 2-2\left(-\tfrac{1}{4}\right)^{n+1}$$

$$4 \overset{\text{tilsv.}}{=} 2 + 3\cdot\left(-\tfrac{1}{4}\right)\cdot\left(-\tfrac{1}{4}\right)^n = 2+3\left(-\tfrac{1}{4}\right)^{n+1}$$

Dermed fås

$$M^{n+1} = \frac{1}{5}\begin{pmatrix} 3+2\left(-\frac{1}{4}\right)^{n+1} & 3-3\left(-\frac{1}{4}\right)^{n+1} \\ 2-2\left(-\frac{1}{4}\right)^{n+1} & 2+3\left(-\frac{1}{4}\right)^{n+1} \end{pmatrix},$$

så formelen holder for $n+1$. Ergo er formelen vist (ved induksjon). Vi får nå

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = M^n \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 3+2\left(-\frac{1}{4}\right)^{n+1} & 3-3\left(-\frac{1}{4}\right)^{n+1} \\ 2-2\left(-\frac{1}{4}\right)^{n+1} & 2+3\left(-\frac{1}{4}\right)^{n+1} \end{bmatrix}\begin{bmatrix} 400 \\ 0 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 1200 + 800\left(-\frac{1}{4}\right)^{n+1} \\ 800 - 800\left(-\frac{1}{4}\right)^{n+1} \end{bmatrix} \xrightarrow{\boxed{n\to\infty}} \frac{1}{5}\begin{bmatrix} 1200 \\ 800 \end{bmatrix} = \begin{bmatrix} 240 \\ 160 \end{bmatrix}$$

når $n \to \infty$. Ergo stabiliserer det seg på omtrent 240 nye og 160 gamle studenter i det lange løp.

Oppgave 16

At $f: \mathbb{R} \to \mathbb{R}$ er kontinuerlig i punktet $x = a$ betyr at det for alle $\varepsilon > 0$ fins $\delta > 0$ slik at

$$|x - a| < \delta \implies |f(x) - f(a)| < \varepsilon.$$

Vi skal vise at $f: \mathbb{R} \to \mathbb{R}$ ved $f(x) = 5x^2$ er kontinuerlig i $x = 1$. Vi har:

Trix: $x = 1 + h$, $h = x - 1$.

$$\begin{aligned} |f(x) - f(1)| &= |5x^2 - 5| \\ &= 5 \cdot |x^2 - 1| \\ &= 5 \cdot |(1+h)^2 - 1| \\ &= 5 \cdot |1 + 2h + h^2 - 1| \\ &= 5 \cdot |h^2 + 2h| \\ &= 5 \cdot |h| \cdot |h + 2| \\ &\leq 15 \cdot |h|, \quad \text{gitt at } |h| \leq 1. \end{aligned}$$

Så: Gitt $\varepsilon > 0$, dersom vi velger δ som det minste av tallene $\varepsilon/15$ og 1, vil vi ha $|f(x) - f(1)| < \varepsilon$ når

$$|x - 1| < \delta.$$