Løsningsforslag eksamen Mat1100 11.12.2015

Fasit del I : CBEAC BBDBB

$\boxed{\text{Oppgave 11}}$

a) $x^2+8x+20 = x^2+8x+16+4$

$$= (x+4)^2+4$$

$$= 4\left[\frac{(x+4)^2}{2^2}+1\right] = 4\left[\left(\frac{x+4}{\sqrt{4}}\right)^2+1\right]$$

Så $\underline{\underline{k=r=4}}$

Ergo

$$\int \frac{1}{(x^2+8x+20)^2}\,dx = \frac{1}{4^2}\int \frac{1}{\left[\left(\frac{x+4}{2}\right)^2+1\right]^2}\,dx$$

$$= \frac{1}{16}\int \frac{1}{(1+u^2)^2}\cdot 2\,du = \frac{2}{16}\int \frac{1}{(1+u^2)^2}\,du$$

$u = \frac{x+4}{2}$ $\quad \frac{du}{dx} = \frac{1}{2}$

$du = \frac{1}{2}dx$ $\quad dx = 2\,du$

$$= \frac{2}{16}\left\{\frac{1}{2}\,\frac{u}{1+u^2} + \frac{2\cdot 2-3}{2\cdot 1}\int \frac{du}{1+u^2}\right\}$$

Oppgitt formel, $m=2$

$$= \frac{1}{16}\left\{\frac{u}{1+u^2} + \arctan u\right\} + C$$

$$= \frac{1}{16}\left\{\frac{\frac{x+4}{2}}{1+\frac{(x+4)^2}{4}} + \arctan \frac{x+4}{2}\right\} + C$$

$$= \frac{1}{16}\left\{\frac{2(x+4)}{4+(x+4)^2} + \arctan \frac{x+4}{2}\right\} + C$$

$$= \frac{1}{16}\left\{\frac{2(x+4)}{x^2+8x+20} + \arctan \frac{x+4}{2}\right\} + C$$

b) $$\int \frac{x+5}{(x^2+8x+20)^2}\,dx = \int \frac{\frac{1}{2}(2x+8)+1}{(x^2+8x+20)^2}\,dx$$

$$= \frac{1}{2}\int \frac{2x+8}{(x^2+8x+20)^2}\,dx + \int \frac{1}{(x^2+8x+20)^2}\,dx$$

$$= \frac{1}{2}\int u^{-2}\,du + [\text{svar a)}]$$

$u = x^2+8x+20$ gir
$du = (2x+8)dx$

$$= -\frac{1}{2}u^{-1} + [\text{svar a)}]$$

$$= -\frac{1}{2(x^2+8x+20)} + [\text{svar a)}]$$

$$= \frac{1}{16}\left[\frac{2x}{x^2+8x+20} + \arctan\frac{x+4}{2}\right] + C$$

$\boxed{\text{Oppgave 12}}$

a) $x_{n+1} = (\text{ant. 0 år ved tid } n+1) = 2 \cdot y_n$

$y_{n+1} = (\text{ant. 1 år ved tid } n+1) = x_n$

$z_{n+1} = (\text{ant. 2 år ved tid } n+1) = y_n$

Altså

$$\begin{cases} x_{n+1} = 0x_n + 2y_n + 0z_n \\ y_{n+1} = 1x_n + 0y_n + 0z_n \\ z_{n+1} = 0x_n + 1y_n + 0z_n \end{cases} \quad \text{dvs.} \quad \begin{pmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix}$$

b)

$$\det M = \begin{vmatrix} 0 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 0 \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} - 2 \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} + 0 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = \underline{\underline{0}}$$

Siden $\det M = 0$, er M ikke inverterbar.

c)

$$M^2 = \begin{array}{ccc|ccc} & & & 0 & 2 & 0 \\ & & & 1 & 0 & 0 \\ & & & 0 & 1 & 0 \\ \hline 0 & 2 & 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{array} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Så

$$M^4 = M^2 \cdot M^2 = \begin{array}{ccc|ccc} & & & 2 & 0 & 0 \\ & & & 0 & 2 & 0 \\ & & & 1 & 0 & 0 \\ \hline 2 & 0 & 0 & 4 & 0 & 0 \\ 0 & 2 & 0 & 0 & 4 & 0 \\ 1 & 0 & 0 & 2 & 0 & 0 \end{array} = \underline{\underline{\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 2 & 0 & 0 \end{bmatrix}}}$$

d) Mønsteret er at

$$M^{2n} = \begin{bmatrix} 2^n & 0 & 0 \\ 0 & 2^n & 0 \\ 2^{n-1} & 0 & 0 \end{bmatrix} \quad \text{for } n = 1, 2, 3, \ldots$$

Vi beviser dette ved induksjon:

$n=1$ Formelen gir $M^{2n} = M^2 = \begin{bmatrix} 2^1 & 0 & 0 \\ 0 & 2^1 & 0 \\ 2^0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

Dette stemmer med tidligere regning.

Anta ok for $n=k$, altså

$$M^{2k} = \begin{bmatrix} 2^k & 0 & 0 \\ 0 & 2^k & 0 \\ 2^{k-1} & 0 & 0 \end{bmatrix} \quad \text{der } k \geq 1 \text{ er et helt tall.}$$

Vi får da

$$M^{2(k+1)} = M^{2k+2} = M^{2k} \cdot M^2 = \begin{array}{ccc|ccc} & & & 2 & 0 & 0 \\ & & & 0 & 2 & 0 \\ & & & 1 & 0 & 0 \\ \hline 2^k & 0 & 0 & 2^{k+1} & 0 & 0 \\ 0 & 2^k & 0 & 0 & 2^{k+1} & 0 \\ 2^{k-1} & 0 & 0 & 2^k & 0 & 0 \end{array}$$

$$= \begin{bmatrix} 2^{k+1} & 0 & 0 \\ 0 & 2^{k+1} & 0 \\ 2^{(k+1)-1} & 0 & 0 \end{bmatrix} \quad \text{Altså ok for } n = k+1.$$

e) Anta først at $n>0$ er et partall. Vi kan da skrive $n = 2k$, der $k \geq 1$ er et helt tall. Dette gir

$$\begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix} = M^n \cdot \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = M^{2k} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \overset{d)}{=} \begin{pmatrix} 2^k & 0 & 0 \\ 0 & 2^k & 0 \\ 2^{k-1} & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2^k \\ 0 \\ 2^{k-1} \end{pmatrix}$$

Så: Hvis $n>0$ er et partall $n = 2k$, har familien

$$2^k + 2^{k-1} = \underline{\underline{2^{n/2} + 2^{(n/2)-1}}}$$ hunkaniner i sesong n

Hvis $n > 1$ er et oddetall, kan vi skrive $n = 2k+1$ der $k > 0$ er et helt tall. Vi får da

$$\begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix} = \begin{pmatrix} x_{2k+1} \\ y_{2k+1} \\ z_{2k+1} \end{pmatrix} = M \cdot \begin{pmatrix} x_{2k} \\ y_{2k} \\ z_{2k} \end{pmatrix} \overset{\text{vet}}{=} \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 2^k \\ 0 \\ 2^{k-1} \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 2^k \\ 0 \end{pmatrix}$$

Så: Hvis $n > 1$ er et oddetall $n = 2k+1$, har familien

$$\underline{\underline{2^k = 2^{(n-1)/2}}}$$

hunkaniner i sesong nummer n. I sesong 1 har familien $\underline{\underline{1}}$ hunkanin.

Slutt!