Fasit til utvalgte oppgaver MAT1110, uka 16-23/1

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Oppgave 2.7.2

Vi setter

$$\begin{aligned} f(u,v) &= ue^{-v} \\ g(x,y,z) &= 2xy+z \\ h(x,y,z) &= 2y(z+x) \\ k(x,y,z) &= f(g(x,y,z),h(x,y,z)) \end{aligned}$$

Vi setter $G(x,y,z) = (g(x,y,z),h(x,y,z))$. Deriverer vi får vi

$$\begin{aligned} f'(u,v) &= \begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \end{pmatrix} = \begin{pmatrix} e^{-v} & -ue^{-v} \end{pmatrix} \\ G'(x,y,z) &= \begin{pmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} & \frac{\partial h}{\partial z} \end{pmatrix} = \begin{pmatrix} 2y & 2x & 1 \\ 2y & 2(z+x) & 2y \end{pmatrix} \end{aligned}$$

Siden $k(x,y,z) = f(G(x,y,z))$ gir kjerneregelen oss at

$$\begin{aligned} k'(x,y,z) &= \begin{pmatrix} \frac{\partial k}{\partial x} & \frac{\partial k}{\partial y} & \frac{\partial k}{\partial z} \end{pmatrix} \\ &= f'(G(x,y,z))G'(x,y,z) = f'(u,v)G'(x,y,z) \\ &= \begin{pmatrix} e^{-v} & -ue^{-v} \end{pmatrix} \begin{pmatrix} 2y & 2x & 1 \\ 2y & 2(z+x) & 2y \end{pmatrix} \\ &= \begin{pmatrix} 2ye^{-v}(1-u) & 2e^{-v}(x-u(z+x)) & e^{-v}(1-2yu) \end{pmatrix} \\ &= (2ye^{-2y(z+x)}(1-2xy-z) \\ &\quad ,2e^{-2y(z+x)}(x-(2xy+z)(z+x)) \\ &\quad ,e^{-2y(z+x)}(1-2y(2xy+z))) \end{aligned}$$

og vi har dermed funnet de partielle deriverte til k.

Oppgave 2.7.6

Vi har at

$$\begin{aligned}\mathbf{H}'(-1,-2,1) &= \mathbf{F}'(2,4)\mathbf{G}'(-1,-2,1) = \begin{pmatrix} 2 & -3 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 4 & -2 & 0 \\ 1 & 3 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 5 & -13 & 3 \\ 2 & 6 & -2 \end{pmatrix}.\end{aligned}$$

Oppgave 2.8.1

Vi ser umidellbart at

$$\begin{aligned} T(\mathbf{e}_1) &= \begin{pmatrix} 2 \\ -1 \end{pmatrix} \\ T(\mathbf{e}_2) &= \begin{pmatrix} -1 \\ 1 \end{pmatrix} \\ T(\mathbf{e}_3) &= \begin{pmatrix} 1 \\ -3 \end{pmatrix} \end{aligned}$$

Vi har derfor at matrisen til T er

$$A = [T(\mathbf{e}_1) \quad T(\mathbf{e}_2) \quad T(\mathbf{e}_3)] = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 1 & -3 \end{pmatrix}.$$

Oppgave 2.8.3

Vi har at

$$\begin{aligned} T(3\mathbf{a}-2\mathbf{b}) &= 3T(\mathbf{a}) - 2T(\mathbf{b}) \\ &= 3\begin{pmatrix} -2 \\ 1 \end{pmatrix} - 2\begin{pmatrix} 0 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} -6 \\ 3 \end{pmatrix} + \begin{pmatrix} 0 \\ -6 \end{pmatrix} = \begin{pmatrix} -6 \\ -3 \end{pmatrix} \end{aligned}$$

Oppgave 2.8.5

Vi har at $T(\mathbf{e}_1) = (1,0)$, $T(\mathbf{e}_2) = (0,2)$. Derfor blir matrisen til T

$$A = [T(\mathbf{e}_1) \quad T(\mathbf{e}_2)] = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

Oppgave 2.8.7

Den lineære avbildningen her sender (x, y, z) på $(x, y, 0)$. Derfor har vi

$$\begin{aligned} T(\mathbf{e}_1) &= (1,0,0) \\ T(\mathbf{e}_2) &= (0,1,0) \\ T(\mathbf{e}_3) &= (0,0,0). \end{aligned}$$

Derfor blir matrisen

$$A = [T(\mathbf{e}_1) \quad T(\mathbf{e}_2) \quad T(\mathbf{e}_3)] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Oppgave 2.8.14

Vi setter $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

a) Vi setter $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Vi får at

$$A\mathbf{v}_1 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1+2 \\ 2+1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 3\mathbf{v}_1.$$

Vi ser derfor at $\mathbf{v}_1$ er en egenvektor for A med tilhørende egenverdi $\lambda_1 = 3$.

b) Vi setter $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Vi får at

$$A\mathbf{v}_2 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1-2 \\ 2-1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = (-1)\begin{pmatrix} 1 \\ -1 \end{pmatrix} = (-1)\mathbf{v}_2.$$

Vi ser derfor at $\mathbf{v}_2$ er en egenvektor for A med tilhørende egenverdi $\lambda_2 = -1$.

c) Vi setter $\mathbf{a} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$. Vi setter

$$\begin{aligned} \mathbf{a} = x\mathbf{v}_1 + y\mathbf{v}_2 \\ \begin{pmatrix} 3 \\ -1 \end{pmatrix} &= x\begin{pmatrix} 1 \\ 1 \end{pmatrix} + y\begin{pmatrix} 1 \\ -1 \end{pmatrix}. \end{aligned}$$

Dette svarer til de to likningene

$$\begin{aligned} x + y &= 3 \\ x - y &= -1, \end{aligned}$$

Legger vi disse sammen, og trekker de fra hverandre får vi først at $2x = 2, 2y = 4$, og deretter $x = 1, y = 2$, slik at $\mathbf{a} = \mathbf{v}_1 + 2\mathbf{v}_2$. Dermed blir

$$\begin{aligned} A^{10}\mathbf{a} &= A^{10}(\mathbf{v}_1 + 2\mathbf{v}_2) = A^{10}\mathbf{v}_1 + 2A^{10}\mathbf{v}_2 \\ &= 3^{10}\mathbf{v}_1 + 2(-1)^{10}\mathbf{v}_2 \\ &= 3^{10}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} 3^{10}+2 \\ 3^{10}-2 \end{pmatrix}. \end{aligned}$$

Oppgave 2.9.1

Vi ser at

$$\begin{aligned} \mathbf{F}(x,y,z) &= \begin{pmatrix} 2x - 3y + z - 7 \\ -x + z - 2 \end{pmatrix} \\ &= \begin{pmatrix} 2x - 3y + z \\ -x + z \end{pmatrix} + \begin{pmatrix} -7 \\ -2 \end{pmatrix} \\ &= \begin{pmatrix} 2 & -3 & 1 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} -7 \\ -2 \end{pmatrix}. \end{aligned}$$

Vi ser derfor at $A = \begin{pmatrix} 2 & -3 & 1 \\ -1 & 0 & 1 \end{pmatrix}$, og at $\mathbf{c} = \begin{pmatrix} -7 \\ -2 \end{pmatrix}$.

Oppgave 2.9.2

Vi ser at

$$\begin{aligned} \mathbf{F}(\mathbf{r}(t)) &= \begin{pmatrix} 1 & -1 & 2 \\ 0 & 3 & -2 \end{pmatrix} + \left(\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + t\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}\right) + \begin{pmatrix} 2 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} 2 + t + 1 + 6 + 4t \\ -3 - 6 - 4t \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} 5t + 9 \\ -4t - 9 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \end{pmatrix} \\ &= t\begin{pmatrix} 5 \\ -4 \end{pmatrix} + \begin{pmatrix} 11 \\ -10 \end{pmatrix}, \end{aligned}$$

som gir oss en parametrisering av $\mathbf{F}(\mathcal{L})$.

Oppgave 2.9.5

Vi har at

$$\begin{aligned}\mathbf{F}(0,0) &= \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ \mathbf{F}(1,0) &= \begin{pmatrix} 2 \\ 3 \end{pmatrix} \\ \mathbf{F}(0,1) &= \begin{pmatrix} -1 \\ 0 \end{pmatrix}\end{aligned}$$

Vi setter nå inn de tre koordinatene i uttrykket

$$\mathbf{F}(x,y) = A\begin{pmatrix} x \\ y \end{pmatrix} + \mathbf{c} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \mathbf{c}$$

og får:

$$\begin{aligned}\mathbf{F}(0,0) &= \mathbf{c} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ \mathbf{F}(1,0) &= \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} + \mathbf{c} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \\ \mathbf{F}(0,1) &= \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} + \mathbf{c} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.\end{aligned}$$

De to siste likningene gir

$$\begin{aligned}\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} &= \begin{pmatrix} 2 \\ 3 \end{pmatrix} - \mathbf{c} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \\ \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} &= \begin{pmatrix} -1 \\ 0 \end{pmatrix} - \mathbf{c} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}.\end{aligned}$$

Vi ser derfor at

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 4 & 1 \end{pmatrix},$$

og derfor er

$$\mathbf{F}(x,y) = A\begin{pmatrix} x \\ y \end{pmatrix} + \mathbf{c} = \begin{pmatrix} 1 & -2 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$