Fasit til utvalgte oppgaver MAT1110, uka 8-12/3

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Oppgave 6.5.5

Vi setter $\mathbf{F}(x,y) = (0, \frac{x^2}{2})$. Skisserer vi kurven ser vi at orienteringen er mot klokka. Dette kan vi også se ved å regne ut

$$\begin{aligned}
\mathbf{v}(t) &= \mathbf{r}'(t) = (1-2t, 1-3t^2) \\
\mathbf{a}(t) &= \mathbf{v}'(t) = (-2, -6t).
\end{aligned}$$

Regner vi ut z-komponenten i $(\mathbf{v}(t), 0) \times (\mathbf{a}(t), 0)$ får vi

$$(1-2t)(-6t) + 2(1-3t^2) = 6t^2 - 6t + 2 = 6\left(t - \frac{1}{2}\right)^2 + \frac{1}{2} > 0.$$

Forsøk så å overbevise deg selv om at fortegnet til z-komponenten i $(\mathbf{v}(t), 0) \times (\mathbf{a}(t), 0)$ avgjør om kurven har positiv eller negativ orientering! Dette trickset er egentlig ikke pensum, så ikke bli oppgitt hvis du ikke forstår det. Det å se orienteringen ut fra en tegning er nok.

Vi regner ut

$$\begin{aligned}
\int\int_R x\,dx\,dy &= \int\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy \\
&= \int_{\mathcal{C}} Q\,dy \\
&= \int_0^1 \frac{(t-t^2)^2}{2}(1-3t^2)dt \\
&= \frac{1}{2}\int_0^1 (t^4 - 2t^3 + t^2)(-3t^2+1)dt \\
&= \frac{1}{2}\int_0^1 (-3t^6 + 6t^5 - 2t^4 - 2t^3 + t^2)dt \\
&= \frac{1}{2}\left[-\frac{3}{7}t^7 + t^6 - \frac{2}{5}t^5 - \frac{1}{2}t^4 + \frac{1}{3}t^3\right]_0^1 \\
&= \frac{1}{2}\left(-\frac{3}{7} + 1 - \frac{2}{5} - \frac{1}{2} + \frac{1}{3}\right) \\
&= \frac{1}{2}\left(\frac{-15-14}{35} + \frac{5}{6}\right) \\
&= \frac{1}{2}\left(\frac{-174+175}{210}\right) = \frac{1}{420}.
\end{aligned}$$

Oppgave 6.5.8

a)

Kurven $\mathcal{C}$ er sammensatt av kurvene $\mathcal{C}_1$ og $\mathcal{C}_2$, der $\mathcal{C}_1$ følger parabelen, $\mathcal{C}_2$ følger langs x-aksen. Vi parametriserer $\mathcal{C}_1$ med $\mathbf{r}_1(x) = (x, 1 - x^2)$, der x går fra 1 til -1. Vi får da

$$\begin{aligned}
\int_{\mathcal{C}_1} -y dx + x^2 dy &= \int_1^{-1} (-(1-x^2) + x^2(-2x))dx \\
&= \int_1^{-1} (-2x^3 + x^2 - 1)dx \\
&= \left[-\frac{1}{2}x^4 + \frac{1}{3}x^3 - x \right]_1^{-1} \\
&= -\frac{1}{3} - \frac{1}{3} + 1 + 1 = \frac{4}{3}.
\end{aligned}$$

Vi parametriserer $\mathcal{C}_2$ med $\mathbf{r}_2(x) = (x, 0)$, der x går fra -1 til 1. Vi får da

$$\int_{\mathcal{C}_2} -y dx + x^2 dy = \int_{-1}^{1} (0 + x^2 \times 0)dx = 0.$$

Vi får derfor

$$\begin{aligned}
\int_{\mathcal{C}} -y dx + x^2 dy &= \int_{\mathcal{C}_1} -y dx + x^2 dy + \int_{\mathcal{C}_2} -y dx + x^2 dy \\
&= \frac{4}{3} + 0 = \frac{4}{3}.
\end{aligned}$$

b)

Med $P(x, y) = -y$ og $Q(x, y) = x^2$ får vi at $\frac{\partial Q}{\partial x} = 2x$, og $\frac{\partial P}{\partial y} = -1$. Parabelen $y = 1 - x^2$ skjærer x-aksen for $x = -1$ og $x = 1$. Vi får dermed

$$\begin{aligned}
\int_{\mathcal{C}} -y dx + x^2 dy &= \int \int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy \\
&= \int_{-1}^{1} \int_0^{1-x^2} (2x + 1) dy dx \\
&= \int_{-1}^{1} (2x + 1)(1 - x^2) dx \\
&= \int_{-1}^{1} (2x - 2x^3 + 1 - x^2) dx \\
&= \left[x^2 - \frac{1}{2}x^4 + x - \frac{1}{3}x^3 \right]_{-1}^{1} \\
&= 1 + 1 - \frac{1}{3} - \frac{1}{3} = \frac{4}{3}.
\end{aligned}$$

Oppgave 6.5.12

a)

Vi fullfører kvadratene og får

$$\begin{aligned}
9x^2 + 4y^2 - 18x + 16y &= 11 \\
9(x^2 - 2x + 1) - 9 + 4(y^2 + 4y + 4) - 16 &= 11 \\
9(x-1)^2 + 4(y+2)^2 &= 36 \\
\frac{(x-1)^2}{2^2} + \frac{(y+2)^2}{3^2} &= 1.
\end{aligned}$$

Vi ser derfor at ellipsen har sentrum $(1, -2)$, store halvakse 3 og lille halvakse 2.

b)

Vi regner ut

$$\begin{aligned}
\frac{(x-1)^2}{2^2} + \frac{(y+2)^2}{3^2} &= \frac{(1 + 2\cos t - 1)^2}{2^2} + \frac{(-2 + 3\sin t + 2)^2}{3^2} \\
&= \frac{(2\cos t)^2}{2^2} + \frac{(3\sin t)^2}{3^2} \\
&= \cos^2 t + \sin^2 t = 1,
\end{aligned}$$

som viser at $\mathbf{r}(t)$ ligger på ellipsen. Det er også klart at $\mathbf{r}(t)$ må dekke hele ellipsen, siden $\mathbf{r}(t)$ vil bevege seg et help omløp mot klokka når t går fra 0 til 2π. Vi har at

$$\begin{aligned}
\int_{\mathcal{C}} \mathbf{F} \cdot dr &= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt \\
&= \int_0^{2\pi} ((-2 + 3\sin t)^2(-2\sin t) + (1 + 2\cos t)(3\cos t)) dt \\
&= \int_0^{2\pi} (-18\sin^3 t + 24\sin^2 t - 8\sin t + 3\cos t + 6\cos^2 t) dt \\
&= \int_0^{2\pi} (24\sin^2 t + 6\cos^2 t) dt = \int_0^{2\pi} (6 + 18\sin^2 t) dt \\
&= \int_0^{2\pi} (6 + 9(1 - \cos(2t))) dt = \int_0^{2\pi} (15 - 9\cos(2t)) dt \\
&= 30\pi.
\end{aligned}$$

c)

Vi ser at

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 - 2y.$$

Vi har derfor at

$$\int\int_R (1 - 2y) dx dy = 30\pi$$

på grunn av Greens teorem og utregningen i b).

Oppgave 6.7.1

a)

Vi har at

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -2.$$

Vi har da at $\frac{\partial(x,y)}{\partial(u,v)} = -\frac{1}{2}$. Integralet blir derfor

$$\begin{aligned}
\int\int_A x^2 dxdy &= \int_0^2 \int_0^1 \frac{1}{2} x^2 dudv \\
&= \int_0^2 \int_0^1 \frac{(v-u)^2}{8} dudv \\
&= \frac{1}{8} \int_0^2 \int_0^1 \left(v^2 - 2uv + u^2\right) dudv \\
&= \frac{1}{8} \int_0^2 \left[uv^2 - u^2 v + \frac{1}{3} u^3 \right]_0^1 dv \\
&= \frac{1}{8} \int_0^2 \left(v^2 - v + \frac{1}{3} \right) dv \\
&= \frac{1}{8} \left[\frac{v^3}{3} - \frac{1}{2} v^2 + \frac{1}{3} v \right]_0^2 \\
&= \frac{1}{8} \left(\frac{8}{3} - 2 + \frac{2}{3} \right) = \frac{1}{6}.
\end{aligned}$$

b)

Vi har at

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1.$$

Integralet blir derfor

$$\begin{aligned}
\int\int_A x dxdy &= \int_0^1 \int_0^3 x dudv \\
&= \int_0^1 \int_0^3 (u+v) dudv \\
&= \int_0^1 \left[\frac{1}{2} u^2 + uv \right]_0^3 dv \\
&= \int_0^1 (\frac{9}{2} + 3v) dv \\
&= \left[\frac{9}{2} v + \frac{3}{2} v^2 \right]_0^1 = \frac{9}{2} + \frac{3}{2} = 6.
\end{aligned}$$

c)

Vi har at

$$\left| \frac{\partial(u,v)}{\partial(x,y)} \right| = \begin{vmatrix} -\frac{1}{2} & 1 \\ -2 & 1 \end{vmatrix} = \frac{3}{2}.$$

Integralgrensene blir $0 \leq u \leq 2$, $-2 \leq v \leq 0$. Integralet blir derfor

$$\begin{aligned}
\int\int_A xydxdy &= \int_{-2}^{0}\int_{0}^{2} xy \left|\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\right| dudv \\
&= \int_{-2}^{0}\int_{0}^{2} \frac{2}{3}(u-v)\frac{1}{3}(4u-v)\frac{2}{3}dudv \\
&= \frac{4}{27}\int_{-2}^{0}\int_{0}^{2}(4u^2-5uv+v^2)dudv \\
&= \frac{4}{27}\int_{-2}^{0}\left[\frac{4}{3}u^3-\frac{5}{2}u^2v+v^2u\right]_0^2 dv \\
&= \frac{4}{27}\int_{-2}^{0}\left(\frac{32}{3}-10v+2v^2\right)dv \\
&= \frac{4}{27}\left[\frac{32}{3}v-5v^2+\frac{2}{3}v^3\right]_{-2}^{0} \\
&= \frac{4}{27}\left(\frac{64}{3}+20+\frac{16}{3}\right) \\
&= \frac{4}{27}\frac{140}{3} = \frac{560}{81}.
\end{aligned}$$

Oppgave 6.7.3

a)

Vi har at

$$\left|\frac{\partial(u,v)}{\partial(x,y)}\right| = \begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} = -3.$$

Derfor blir $\left|\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\right| = \frac{1}{3}$. Videre blir integralgrensene $-1 \leq u \leq 3$, $1 \leq v \leq 4$. Integralet blir derfor

$$\begin{aligned}
\int\int_R xydxdy &= \int_{-1}^{3}\int_{1}^{4} xy \left|\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\right| dvdu \\
&= \int_{-1}^{3}\int_{1}^{4} \frac{1}{3}(u+2v)\frac{1}{3}(u-v)\frac{1}{3}dvdu \\
&= \frac{1}{27}\int_{-1}^{3}\int_{1}^{4}(u^2+uv-2v^2)dvdu \\
&= \frac{1}{27}\int_{-1}^{3}\left[u^2v+\frac{1}{2}uv^2-\frac{2}{3}v^3\right]_1^4 du \\
&= \frac{1}{27}\int_{-1}^{3}\left(3u^2+\frac{15}{2}u-42\right)du \\
&= \frac{1}{27}\left[u^3+\frac{15}{4}u^2-42u\right]_{-1}^{3} \\
&= \frac{1}{27}(28+30-168) \\
&= -\frac{110}{27}.
\end{aligned}$$

For å beregne integralet med Matlab eller Python kan vi først finne skjæringen mellom de fire linjene som definerer området:

- Skjæring mellom $x+2y=-1$ og $x-y=1$: $(x,y)=(\frac{1}{3},-\frac{2}{3})$.
- Skjæring mellom $x+2y=-1$ og $x-y=4$: $(x,y)=(\frac{7}{3},-\frac{5}{3})$.
- Skjæring mellom $x+2y=3$ og $x-y=1$: $(x,y)=(\frac{5}{3},\frac{2}{3})$.
- Skjæring mellom $x+2y=3$ og $x-y=4$: $(x,y)=(\frac{11}{3},-\frac{1}{3})$.

Integrasjonsområdet ligger derfor innenfor rektanglet $\frac{1}{3}\leq x\frac{11}{3}$, $-\frac{5}{3}\leq y\leq\frac{2}{3}$.

c)

Vi har at

$$\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix} y & x \\ -\frac{y}{x^2} & \frac{1}{x}\end{vmatrix}=\frac{y}{x}+\frac{y}{x}=\frac{2y}{x}=2v.$$

Derfor blir $\frac{\partial(x,y)}{\partial(u,v)}=\frac{1}{2v}$. Integralet blir derfor

$$\begin{aligned}
\int\int_R (y^2-yx)dxdy &= \int_1^2\int_1^2 (y^2-yx)\frac{1}{2v}dvdu \\
&= \int_1^2\int_1^2 \frac{uv-u}{2v}dvdu \\
&= \int_1^2\int_1^2 \left(\frac{u}{2}-\frac{u}{2v}\right)dvdu \\
&= \int_1^2 \left[\frac{u}{2}(v-\ln v)\right]_1^2 du \\
&= \int_1^2 \frac{u}{2}(1-\ln 2)du \\
&= \left[\frac{u^2}{4}(1-\ln 2)\right]_1^2=\frac{3}{4}(1-\ln 2).
\end{aligned}$$

Oppgave 6.7.5

a)

Vi setter $u=y-x$, $v=y+x$. Vi har at

$$\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix} -1 & 1 \\ 1 & 1\end{vmatrix}=-2.$$

Derfor blir $\frac{\partial(x,y)}{\partial(u,v)}=-\frac{1}{2}$. Integralet blir derfor

$$\begin{aligned}
\int\int_A \frac{e^{x-y}}{x+y}dxdy &= \int_0^5\int_2^4 \frac{e^{-u}}{2v}dvdu \\
&= \int_0^5 \left[\frac{1}{2}e^{-u}\ln v\right]_2^4 du \\
&= \int_0^5 \frac{1}{2}e^{-u}(\ln 4-\ln 2)du=\int_0^5\frac{1}{2}e^{-u}\ln 2du \\
&= \left[-\frac{1}{2}e^{-u}\ln 2\right]_0^5=-\frac{1}{2}e^{-5}\ln 2+\frac{1}{2}\ln 2=\frac{1}{2}\ln 2(1-e^{-5}).
\end{aligned}$$

b)

Vi setter $u = \frac{y}{x}$, $v = yx$. Vi har at

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} -\frac{y}{x^2} & \frac{1}{x} \\ y & x \end{vmatrix} = -\frac{y}{x} - \frac{y}{x} = -\frac{2y}{x} = -2u.$$

Derfor blir $\frac{\partial(x,y)}{\partial(u,v)} = -\frac{1}{2u}$. Integralet blir derfor

$$\begin{aligned} \int\int_A xydxdy &= \int_1^2 \int_1^3 \frac{v}{2u} dvdu \\ &= \int_1^2 \left[\frac{1}{4u}v^2\right]_1^3 du = \int_1^2 \frac{2}{u} du = [2\ln u]_1^2 = 2\ln 2. \end{aligned}$$

Oppgave 6.7.8

a)

Vi kan begrense oss til verdier $u > 0$, $0 \leq v \leq 2\pi$.

- Første kvadrant av xy-planet svarer til at $0 \leq v \leq \frac{\pi}{2}$.
- $y = 2x$ svarer til at $2u \sin v = 2u \cos v$, slik at $v = \frac{\pi}{4}$ eller $v = \frac{5\pi}{4}$. Her er det bare den første vi er interessert i.
- Ellipsen $x^2 + \frac{y^2}{4} = 1$ svarer til at $u^2 \cos^2 v + \frac{4u^2 \sin^2 v}{4} = u^2 = 1$, slik at $u = 1$ eller $u = -1$. Området innenfor ellipsen ser vi derfor er beskrevet ved at $0 \leq u \leq 1$.

Vi ser at området vårt, D, er beskrevet ved at $0 \leq u \leq 1$, $0 \leq v \leq \frac{\pi}{4}$. La så $\mathbf{T}(u,v) = (u\cos v, 2u \sin v)$. Vi har at

$$\begin{aligned} \mathbf{T}'(u,v) &= \begin{pmatrix} \cos v & -u\sin v \\ 2\sin v & 2u\cos v \end{pmatrix} \\ |\det \mathbf{T}'(u,v)| &= |2u\cos^2 v + 2u\sin^2 v| = 2u. \end{aligned}$$

Arealet blir derfor

$$\begin{aligned} \int\int_R dxdy &= \int\int_{\mathbf{T}(D)} dxdy \\ & \int\int_D |\det \mathbf{T}'(u,v)| dudv = \int_0^{\frac{\pi}{4}} \int_0^1 |\det \mathbf{T}'(u,v)| dudv \\ &= \int_0^{\frac{\pi}{4}} \int_0^1 2ududv = \int_0^{\frac{\pi}{4}} \left[u^2\right]_0^1 dv \\ &= \int_0^{\frac{\pi}{4}} dv = \frac{\pi}{4}. \end{aligned}$$

b)

Flaten $z = x^2 + \frac{y^2}{2} = u^2 \cos^2 v + 2u^2 \sin^2 v = u^2(1 + \sin^2 v)$ kan parametriseres ved hjelp av u og v ved

$$\mathbf{r}(u,v) = (u\cos v, 2u\sin v, u^2(1 + \sin^2 v))$$

Vi får da

$$\begin{aligned}
\frac{\partial r}{\partial u} &= (\cos v, 2\sin v, 2u(1+\sin^2 v)) \\
\frac{\partial r}{\partial v} &= (-u\sin v, 2u\cos v, 2u^2\sin v\cos v) \\
\frac{\partial r}{\partial u}\times\frac{\partial r}{\partial v} &= (-4u^2\cos v, -4u^2\sin v, 2u) \\
\left|\frac{\partial r}{\partial u}\times\frac{\partial r}{\partial v}\right| &= \sqrt{16u^4+4u^2} = 2u\sqrt{4u^2+1}.
\end{aligned}$$

Arealet av flaten er gitt ved

$$\begin{aligned}
\int\int_D \left|\frac{\partial r}{\partial u}\times\frac{\partial r}{\partial v}\right| dudv &= \int_0^{\frac{\pi}{4}}\int_0^1 2u\sqrt{4u^2+1}dudv \\
&= \int_0^{\frac{\pi}{4}}\left[\frac{1}{6}(4u^2+1)^{3/2}\right]_0^1 dudv \\
&= \frac{1}{6}\int_0^{\frac{\pi}{4}}(5^{3/2}-1)dv \\
&= \frac{1}{6}\frac{\pi}{4}(5\sqrt{5}-1) = \frac{\pi(5\sqrt{5}-1)}{24}.
\end{aligned}$$

Oppgave 6.8.1

Området mellom x-aksen og linjen $y=x$ i første kvadrant er beskrevet i polarkoordinater ved $0\le\theta\le\frac{\pi}{4}$. Vi får derfor

$$\begin{aligned}
\int\int_A e^{-x^2-y^2}dxdy &= \lim_{n\to\infty}\int\int_{A\cap B(0,n)} e^{-x^2-y^2}dxdy \\
&= \lim_{n\to\infty}\int\int_{A\cap B(0,n)} e^{-r^2}rdrd\theta \\
&= \lim_{n\to\infty}\int_0^{\pi/4}\int_0^n e^{-r^2}rdrd\theta \\
&= \lim_{n\to\infty}\int_0^{\pi/4}\left[-\frac{1}{2}e^{-r^2}\right]\int_0^n d\theta \\
&= \lim_{n\to\infty}\int_0^{\pi/4}\left(\frac{1}{2}-\frac{1}{2}e^{-n^2}\right)d\theta \\
&= \lim_{n\to\infty}\frac{\pi}{4}\left(\frac{1}{2}-\frac{1}{2}e^{-n^2}\right) \\
&= \frac{\pi}{4}\frac{1}{2} = \frac{\pi}{8}.
\end{aligned}$$

Oppgave 6.8.2

$$\begin{aligned}
\int\int_{\mathbb{R}^2} \frac{1}{1+x^2+y^2} dxdy &= \lim_{n\to\infty} \int\int_{B(0,n)} \frac{1}{1+x^2+y^2} dxdy \\
&= \lim_{n\to\infty} \int_0^{2\pi} \int_0^n \frac{r}{1+r^2} drd\theta \\
&= \lim_{n\to\infty} \int_0^{2\pi} \int_0^n \left[\frac{1}{2}\ln(1+r^2)\right]_0^n d\theta \\
&= \lim_{n\to\infty} \int_0^{2\pi} \int_0^n \frac{1}{2}\ln(1+n^2) d\theta \\
&= \lim_{n\to\infty} \pi \ln(1+n^2) = \infty.
\end{aligned}$$

Derfor divergerer integralet.

Oppgave 6.8.4

Det er klart at $f(x,y) = xy$ er en positiv funksjon på A, siden A er inneholdt i første kvadrant. Siden

$$A \cap K_n = \{(x,y)|\frac{1}{n} \leq x \leq n, 0 \leq y \leq \frac{1}{x}\} \cup \{(x,y)|0 \leq x \leq \frac{1}{n}, 0 \leq y \leq n\}$$

så splitter vi integralet i to biter:

$$\begin{aligned}
\int\int_A f(x,y)dxdy &= \lim_{n\to\infty} \int\int_{A\cap K_n} xydxdy \\
&= \lim_{n\to\infty} \int_0^{\frac{1}{n}} \int_0^n xydxdy + \int_{\frac{1}{n}}^n \int_0^{\frac{1}{x}} xydydx \\
&= \lim_{n\to\infty} \left(\int_0^{\frac{1}{n}} \left[\frac{1}{2}xy^2\right]_0^n dx + \int_{\frac{1}{n}}^n \left[\frac{1}{2}xy^2\right]_0^{\frac{1}{x}} dx \right) \\
&= \lim_{n\to\infty} \left(\int_0^{\frac{1}{n}} \frac{1}{2}xn^2 dx + \int_{\frac{1}{n}}^n \frac{1}{2x} dx \right) \\
&= \lim_{n\to\infty} \left(\left[\frac{n^2}{4}x^2\right]_0^{\frac{1}{n}} + \left[\frac{1}{2}\ln(x)\right]_{\frac{1}{n}}^n \right) \\
&= \lim_{n\to\infty} \left(\frac{1}{4} + \frac{1}{2}\ln n - \frac{1}{2}\ln(\frac{1}{n}) \right) \\
&= \lim_{n\to\infty} \left(\frac{1}{4} + \ln n \right) = \infty.
\end{aligned}$$

Integralet konvergerer derfor ikke på A.

Oppgave 6.8.5

Det er her lurt å integrere med tanke på x først, siden integralgrensene da blir enklest. Definerer vi $A_n = \{(x,y)|0 \le y \le n, 0 \le x \le \sqrt{y}\}$ får vi at Vi har at

$$\begin{aligned}
\int_A \frac{x}{1+y^4}dydx &= \lim_{n\to\infty}\int\int_{A_n}\frac{x}{1+y^4}dxdy \\
&= \lim_{n\to\infty}\int_0^n\int_0^{\sqrt{y}}\frac{x}{1+y^4}dxdy \\
&= \lim_{n\to\infty}\int_0^n\left[\frac{x^2}{2(1+y^4)}\right]_0^{\sqrt{y}}dy \\
&= \lim_{n\to\infty}\int_0^n\frac{y}{2(1+y^4)}dy \\
&= \lim_{n\to\infty}\left[\frac{1}{4}\arctan(y^2)\right]_0^n \\
&= \frac{1}{4}\lim_{n\to\infty}\arctan(n^2) \\
&= \frac{\pi}{8}.
\end{aligned}$$

Oppgave 6.9.1

a)

$$\begin{aligned}
\int\int\int_A xyzdxdydz &= \int_0^1\left[\int_0^1\left[\int_0^1 xyzdx\right]dy\right]dz \\
&= \int_0^1\left[\int_0^1[\frac{1}{2}x^2yz]_0^1dy\right]dz = \int_0^1\left[\int_0^1\frac{1}{2}yzdy\right]dz \\
&= \int_0^1[\frac{1}{4}y^2z]_0^1dz = \int_0^1\frac{1}{4}zdz \\
&= [\frac{1}{8}z^2]_0^1 = \frac{1}{8}.
\end{aligned}$$

b)

$$\begin{aligned}
\int\int\int_A (x+ye^z)dxdydz &= \int_{-1}^1\left[\int_0^1\left[\int_1^2(x+ye^z)dz\right]dy\right]dx \\
&= \int_{-1}^1\left[\int_0^1[xz+ye^z]_1^2\,dy\right]dx \\
&= \int_{-1}^1\left[\int_0^1\big(x+y(e^2-e)\big)\,dy\right]dx \\
&= \int_{-1}^1\left[xy+\frac{1}{2}(e^2-e)y^2\right]_0^1dx \\
&= \int_{-1}^1\left(x+\frac{1}{2}(e^2-e)\right)dx \\
&= \left[\frac{1}{2}x^2+\frac{1}{2}(e^2-e)x\right]_{-1}^1 \\
&= \frac{1}{2}+\frac{1}{2}(e^2-e)-\frac{1}{2}+\frac{1}{2}(e^2-e) = e^2-e.
\end{aligned}$$

Oppgave 6.9.2

a)

$$\begin{aligned}
\int\int\int_A (xy+z)dxdydz &= \int_0^1 \left[\int_0^2 \left[\int_0^{x^2y} (xy+z)dz\right] dy\right] dx \\
&= \int_0^1 \left[\int_0^2 \left[xyz + \frac{1}{2}z^2\right]_0^{x^2y} dy\right] dx \\
&= \int_0^1 \left[\int_0^2 \left(x^3y^2 + \frac{1}{2}x^4y^2\right) dy\right] dx \\
&= \int_0^1 \left[\frac{1}{3}\left(x^3 + \frac{1}{2}x^4\right) y^3\right]_0^2 dx \\
&= \int_0^1 \left(\frac{8}{3}x^3 + \frac{4}{3}x^4\right) dx \\
&= \left[\frac{2}{3}x^4 + \frac{4}{15}x^5\right]_0^1 \\
&= \frac{2}{3} + \frac{4}{15} = \frac{10+4}{15} = \frac{14}{15}.
\end{aligned}$$

b)

$$\begin{aligned}
\int\int\int_A zdxdydz &= \int_0^2 \left[\int_0^{\sqrt{x}} \left[\int_{-y^2}^{xy} zdz\right] dy\right] dx \\
&= \int_0^2 \left[\int_0^{\sqrt{x}} \left[\frac{1}{2}z^2\right]_{-y^2}^{xy} dy\right] dx \\
&= \int_0^2 \left[\int_0^{\sqrt{x}} \left(\frac{1}{2}x^2y^2 - \frac{1}{2}y^4\right) dy\right] dx \\
&= \int_0^2 \left[\frac{1}{6}x^2y^3 - \frac{1}{10}y^5\right]_0^{\sqrt{x}} dx \\
&= \int_0^2 \left(\frac{1}{6}x^{7/2} - \frac{1}{10}x^{5/2}\right) dx \\
&= \left[\frac{1}{27}x^{9/2} - \frac{1}{35}x^{7/2}\right]_0^2 \\
&= \frac{1}{27}2^{9/2} - \frac{1}{35}2^{7/2} \\
&= \frac{16\sqrt{2}}{27} - \frac{8\sqrt{2}}{35} = 8\sqrt{2}\left(\frac{2}{27} - \frac{1}{35}\right) = \frac{344\sqrt{2}}{945}.
\end{aligned}$$

c)

$$\begin{aligned}
\int\int\int_A (x+y)z\,dxdydz &= \int_0^4 \left[\int_0^{\sqrt{y}} \left[\int_0^4 (x+y)zdz\right] dx\right] dy \\
&= \int_0^4 \left[\int_0^{\sqrt{y}} \left[\frac{1}{2}(x+y)z^2\right]_0^4 dx\right] dy \\
&= \int_0^4 \left[\int_0^{\sqrt{y}} 8(x+y)dx\right] dy \\
&= \int_0^4 \left[4x^2+8xy\right]_0^{\sqrt{y}} dy \\
&= \int_0^4 \left(4y+8y^{3/2}\right) dy \\
&= \left[2y^2+\frac{16}{5}y^{5/2}\right]_0^4 \\
&= 32+\frac{16}{5}32 = 32\frac{21}{5} = \frac{672}{5}.
\end{aligned}$$

e)

Pyramiden kan beskrives ved $0 \leq x \leq 1$, $0 \leq y \leq 1-x$, $0 \leq z \leq 1-x-y$. Vi får

$$\begin{aligned}
\int\int\int_A xydxdydz &= \int_0^1 \left[\int_0^{1-x} \left[\int_0^{1-x-y} xydz\right] dy\right] dx \\
&= \int_0^1 \left[\int_0^{1-x} [xyz]_0^{1-x-y}\, dy\right] dx \\
&= \int_0^1 \left[\int_0^{1-x} xy(1-x-y)dy\right] dx \\
&= \int_0^1 \left[\int_0^{1-x} (xy-x^2y-xy^2)dy\right] dx \\
&= \int_0^1 \left[\frac{1}{2}xy^2-\frac{1}{2}x^2y^2-\frac{1}{3}xy^3\right]_0^{1-x} dx \\
&= \int_0^1 \left(\frac{1}{2}x(1-x)^2-\frac{1}{2}x^2(1-x)^2-\frac{1}{3}x(1-x)^3\right) dx \\
&= \int_0^1 \left(\frac{1}{2}x(1-x)^3-\frac{1}{3}x(1-x)^3\right) dx \\
&= \int_0^1 \frac{1}{6}x(1-x)^3dx \\
&= \frac{1}{6}\int_0^1 (x-3x^2+3x^3-x^4)dx \\
&= \frac{1}{6}\left[\frac{1}{2}x^2-x^3+\frac{3}{4}x^4-\frac{1}{5}x^5\right]_0^1 \\
&= \frac{1}{6}\left(\frac{1}{2}-1+\frac{3}{4}-\frac{1}{5}\right) \\
&= \frac{1}{6}\frac{10-20+15-4}{20} = \frac{1}{120}.
\end{aligned}$$

Oppgave 6.10.1

a)

$0 \le x, y \le 1$ betyr i sylinderkoordinater $0 \le \theta \le \frac{\pi}{2}$. $x^2 + y^2 \le 9$ betyr $r \le 3$.

$$\begin{aligned}
\int\int\int_A x dx dy dz &= \int_0^{\pi/2} \left[\int_0^3 \left[\int_0^2 r^2 \cos\theta dz \right] dr \right] d\theta \\
&= \int_0^{\pi/2} \left[\int_0^3 2r^2 \cos\theta dr \right] d\theta \\
&= \int_0^{\pi/2} \left[\frac{2}{3} r^3 \cos\theta \right]_0^3 d\theta \\
&= \int_0^{\pi/2} 18 \cos\theta d\theta \\
&= [18 \sin\theta]_0^{\pi/2} \\
&= 18.
\end{aligned}$$

b)

$$\begin{aligned}
\int\int\int_A xy dx dy dz &= \int_0^{2\pi} \left[\int_0^1 \left[\int_0^{4-r(\cos\theta+\sin\theta)} r^3 \sin\theta\cos\theta dz \right] dr \right] d\theta \\
&= \int_0^{2\pi} \left[\int_0^1 \left[zr^3 \sin\theta\cos\theta \right]_0^{4-r(\cos\theta+\sin\theta)} dr \right] d\theta \\
&= \int_0^{2\pi} \left[\int_0^1 (4 - r(\cos\theta + \sin\theta)) r^3 \sin\theta\cos\theta dr \right] d\theta \\
&= \int_0^{2\pi} \left[\int_0^1 \left(2r^3 \sin(2\theta) - r^4 (\sin^2\theta\cos\theta + \sin\theta\cos^2\theta) \right) dr \right] d\theta \\
&= \int_0^{2\pi} \left[\frac{1}{2} r^4 \sin(2\theta) - \frac{1}{5} r^5 (\sin^2\theta\cos\theta + \sin\theta\cos^2\theta) \right]_0^1 d\theta \\
&= \int_0^{2\pi} \left(\frac{1}{2} \sin(2\theta) - \frac{1}{5} (\sin^2\theta\cos\theta + \sin\theta\cos^2\theta) \right) d\theta \\
&= \left[-\frac{1}{4} \cos(2\theta) - \frac{1}{5} \left(\frac{1}{3} \sin^3\theta - \frac{1}{3} \cos^3\theta \right) \right]_0^{2\pi} \\
&= 0.
\end{aligned}$$

Oppgave 6.10.2

a)

Vi setter inn $x^2 + y^2 = \rho^2 \sin^2 \phi \cos^2 \theta + \rho^2 \sin^2 \phi \sin^2 \theta = \rho^2 \sin^2 \phi$, og Jacobideterminanten $\rho^2 \sin \phi$ og får

$$\begin{aligned}
\int\int\int_A (x^2 + y^2) dxdydz &= \int_0^{2\pi} \left[\int_0^{\pi} \left[\int_0^1 \rho^2 \sin^2 \phi \rho^2 \sin \phi d\rho \right] d\phi \right] d\theta \\
&= \int_0^{2\pi} \left[\int_0^{\pi} \left[\int_0^1 \rho^4 \sin^3 \phi d\rho \right] d\phi \right] r\theta \\
&= \int_0^{2\pi} \left[\int_0^{\pi} \left[\frac{1}{5} \rho^5 \sin^3 \phi \right]_0^1 d\phi \right] d\theta \\
&= \int_0^{2\pi} \left[\int_0^{\pi} \frac{1}{5} \sin^3 \phi d\phi \right] d\theta \\
&= \int_0^{2\pi} \left[\int_0^{\pi} \frac{1}{5} (1 - \cos^2 \phi) \sin \phi d\phi \right] d\theta \\
&= \int_0^{2\pi} \left[\frac{1}{5} (- \cos \phi + \frac{1}{15} \cos^3 \phi) \right]_0^{\pi} d\theta \\
&= \int_0^{2\pi} \left(\frac{1}{5} - \frac{1}{15} + \frac{1}{5} - \frac{1}{15} \right) d\theta \\
&= \int_0^{2\pi} \frac{4}{15} d\theta \\
&= \frac{8\pi}{15}.
\end{aligned}$$

b)

$0 \le x, y \le 1$ betyr i kulekoordinater at $0 \le \theta \le \frac{\pi}{2}$. $x^2+y^2+z^2 \le 1$ betyr at $\rho \le 1$. Kombinert med $z \ge \frac{1}{2}$ betyr dette at $\frac{1}{2\cos\phi} \le \rho \le 1$. Vi får derfor

$$\begin{aligned}
\int\int\int_A x dx dy dz &= \int_0^{\pi/2}\left[\int_0^{\pi/3}\left[\int_{\frac{1}{2\cos\phi}}^1 \rho\sin\phi\cos\theta\rho^2\sin\phi d\rho\right]d\phi\right]d\theta \\
&= \int_0^{\pi/2}\left[\int_0^{\pi/3}\left[\int_{\frac{1}{2\cos\phi}}^1 \rho^3\sin^2\phi\cos\theta d\rho\right]d\phi\right]d\theta \\
&= \int_0^{\pi/2}\left[\int_0^{\pi/3}\left[\frac{1}{4}\rho^4\sin^2\phi\cos\theta\right]_{\frac{1}{2\cos\phi}}^1 d\phi\right]d\theta \\
&= \int_0^{\pi/2}\left[\int_0^{\pi/3}\cos\theta\left(\frac{1}{4}\sin^2\phi - \frac{1}{64}\frac{\sin^2\phi}{\cos^4\phi}\right)d\phi\right]d\theta \\
&= \int_0^{\pi/2}\left[\int_0^{\pi/3}\cos\theta\left(\frac{1}{8}(1-\cos(2\phi)) - \frac{1}{64}\frac{\tan^2\phi}{\cos^2\phi}\right)d\phi\right]d\theta \\
&= \int_0^{\pi/2}\left[\cos\theta\left(\frac{1}{8}\phi - \frac{1}{16}\sin(2\phi) - \frac{1}{196}\tan^3\phi\right)\right]_0^{\pi/3}d\theta \\
&= \int_0^{\pi/2}\cos\theta\left(\frac{\pi}{24} - \frac{\sqrt{3}}{32} - \frac{3\sqrt{3}}{196}\right)d\theta \\
&= \int_0^{\pi/2}\cos\theta\left(\frac{\pi}{24} - \frac{3\sqrt{3}}{64}\right)d\theta \\
&= \frac{\pi}{24} - \frac{3\sqrt{3}}{64},
\end{aligned}$$

hvor subsitusjonen $u = \tan\phi$ ($du = \frac{d\phi}{\cos^2\phi}$) ble brukt.

Oppgave 6.10.3

a)

Vi finner først skjæringspunktene mellom paraboloiden og kuleflaten:

$$x^2+y^2 = \sqrt{2-x^2-y^2} \iff r^2 = \sqrt{2-r^2} \iff r^4 = 2-r^2 \iff r = \frac{-1 \pm \sqrt{1+8}}{2}.$$

Eneste positive løsning her er $r = 1$. Vi får derfor (området er beskrevet ved $0 \le \theta \le 2\pi$ og $0 \le r \le 1$, og kuleflaten ligger øverst)

$$\begin{aligned}
\int\int\int_A z dx dy dz &= \int_0^{2\pi}\left[\int_0^1\left[\int_{r^2}^{\sqrt{2-r^2}} zr dz\right]dr\right]d\theta \\
&= \int_0^{2\pi}\left[\int_0^1\left[\frac{1}{2}z^2 r\right]_{r^2}^{\sqrt{2-r^2}} dr\right]d\theta \\
&= \int_0^{2\pi}\left[\int_0^1\left(\frac{1}{2}(2-r^2)r - \frac{1}{2}r^5\right)dr\right]d\theta \\
&= \int_0^{2\pi}\left[\int_0^1\left(-\frac{1}{2}r^5 - \frac{1}{2}r^3 + r\right)dr\right]d\theta \\
&= \int_0^{2\pi}\left[-\frac{1}{12}r^6 - \frac{1}{8}r^4 + \frac{1}{2}r^2\right]_0^1 d\theta \\
&= \int_0^{2\pi}\left(-\frac{1}{12} - \frac{1}{8} + \frac{1}{2}\right)d\theta \\
&= \int_0^{2\pi}\frac{-2-3+12}{24}d\theta \\
&= \frac{7\pi}{12}.
\end{aligned}$$

b)

$$\begin{aligned}
\int\int\int_A x dx dy dz &= \int_0^{2\pi}\left[\int_0^2\left[\int_{r^2}^4 r^2\cos\theta dz\right]dr\right]d\theta \\
&= \int_0^{2\pi}\left[\int_0^2 \left[r^2\cos\theta z\right]_{r^2}^4 dr\right]d\theta \\
&= \int_0^{2\pi}\left[\int_0^2 \left(4r^2\cos\theta - r^4\cos\theta\right)dr\right]d\theta \\
&= \int_0^{2\pi}\left[\frac{4}{3}r^3\cos\theta - \frac{1}{5}r^5\cos\theta\right]_0^2 d\theta \\
&= \int_0^{2\pi}\left(\frac{32}{3} - \frac{32}{5}\right)\cos\theta d\theta \\
&= 32\int_0^{2\pi}\frac{2}{15}\cos\theta d\theta \\
&= \frac{64}{15}\left[\sin\theta\right]_0^{2\pi} = 0.
\end{aligned}$$

e)

Likningen $x^2 - 2x + y^2 = 1$ kan skrives $(x-1)^2 + y^2 = 2$, som er en sirkel med sentrum i $(1,0)$ med radius $\sqrt{2}$. Vi setter $u = x-1, v = y, w = z$, og ser umiddelbart at Jacobideterminanten blir 1. Lar vi D være den delen av sylinderen $u^2 + v^2 = 2$

som ligger mellom planene $z=0$ og $z=2$ får vi

$$\begin{aligned}
\int\int\int_A (x^2+y^2)dxdydz &= \int\int\int_D ((u+1)^2+v^2)dudv \\
&= \int_0^{2\pi}\left[\int_0^{\sqrt{2}}\left[\int_0^2 ((u+1)^2+v^2)rdz\right]dr\right]d\theta \\
&= \int_0^{2\pi}\left[\int_0^{\sqrt{2}}\left[\int_0^2 ((r\cos\theta+1)^2+r^2\sin^2\theta)rdz\right]dr\right]d\theta \\
&= \int_0^{2\pi}\left[\int_0^{\sqrt{2}}\left[\int_0^2 (r^3+2r^2\cos\theta+r)dz\right]dr\right]d\theta \\
&= \int_0^{2\pi}\left[\int_0^{\sqrt{2}} (2r^3+4r^2\cos\theta+2r)dr\right]d\theta \\
&= \int_0^{2\pi}\left[\frac{1}{2}r^4+\frac{4}{3}r^3\cos\theta+r^2\right]_0^{\sqrt{2}}d\theta \\
&= \int_0^{2\pi}\left(2+\frac{4}{3}2\sqrt{2}\cos\theta+2\right)d\theta \\
&= \left[4\theta+\frac{8}{3}\sqrt{2}\sin\theta\right]_0^{2\pi} = 8\pi.
\end{aligned}$$

Oppgave 6.10.5

Vi bruker kulekoordinater. Siden kjeglen $z=\sqrt{x^2+y^2}=\rho\sin\phi=\rho\cos\phi$ så må $\phi=\frac{\pi}{4}$ på kjeglen.

$$\begin{aligned}
\int\int\int_D zdxdydz &= \int_0^{2\pi}\left[\int_0^{\pi/4}\left[\int_0^1 \rho\cos\phi\rho^2\sin\phi d\rho\right]d\phi\right]d\theta \\
&= \int_0^{2\pi}\left[\int_0^{\pi/4}\left[\int_0^1 \frac{1}{2}\rho^3\sin(2\phi)d\rho\right]d\phi\right]d\theta \\
&= \int_0^{2\pi}\left[\int_0^{\pi/4}\left[\frac{1}{8}\rho^4\sin(2\phi)\right]_0^1 d\phi\right]d\theta \\
&= \int_0^{2\pi}\left[\int_0^{\pi/4}\frac{1}{8}\sin(2\phi)d\phi\right]d\theta \\
&= \int_0^{2\pi}\left[-\frac{1}{16}\cos(2\phi)\right]_0^{\pi/4}d\theta \\
&= \int_0^{2\pi}\frac{1}{16}d\theta = \frac{\pi}{8}.
\end{aligned}$$

Matlab-kode

```
% Oppgave 6.7.2 a)
dblquad(@(x,y)x.^2.*(x<=y).*(y<=x+1).*(-x<=y).*(y<=-x+2),-0.5,1,0,1.5)

% Oppgave 6.7.2 b)
dblquad(@(x,y)x.*(y<=x).*(x-3<=y),0,4,0,1)
```

```
% Oppgave 6.7.2 c)
dblquad(@(x,y)x.*y.*(y<=2*x).*(y<=(x/2)+2).*(2*x-2<=y).*(x/2<=y),0,8/3,0,10/3)
```

Python-kode

```
from integrate2D import *

# Oppgave 6.7.2 a)
print integrate2D(lambda x,y: x**2*(x<=y)*(y<=x+1)*(-x<=y)*(y<=-x+2),-0.5,1,0,1.5,100,100)

# Oppgave 6.7.2 b)
print integrate2D(lambda x,y: x*(y<=x)*(x-3<=y),0,4,0,1,100,100)

# Oppgave 6.7.2 c)
print integrate2D(lambda x,y: x*y*(y<=2*x)*(y<=(x/2)+2)*(2*x-2<=y)*(x/2<=y),0,8.0/3,0,10.0/3,100,100)
```