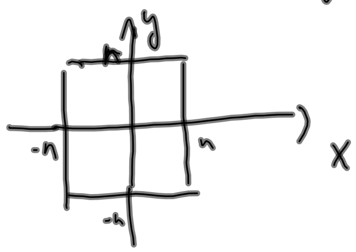
6.8

$$\iint_A f(x,y)\,dx\,dy$$

A lukket, begrenset
Jordan målbar
f kontinuerlig

$K_n = \{ (x,y) \mid |x|, |y| \le n \}$



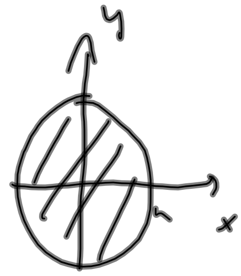
Hvis $A \subseteq \mathbb{R}^2$
og f er ikke negativ
og $A \cap K_n$ er lukket, begrenset og Jordan målbar
og f er kontinuerlig på A, f

Dersom $\lim_{n\to\infty} \iint_{A \cap K_n} f(x,y)\,dx\,dy = L$ eksisterer

sier vi at det uegentlige integralet $\iint_A f(x,y)\,dx\,dy$ konvergerer mot L, dersom

grensen ikke fins, så sier vi at det uegentlige integralet divergerer.

$$\lim_{n\to\infty} \iint_{A\cap K_n} f(x,y)\,dx\,dy.$$



$$B_{(0,n)} = \{(x,y) \mid x^2+y^2 \le n^2\}$$

<u>Setning</u>: Dersom $\iint_A f(x,y)\,dx\,dy = L$ konvergerer så fins grensen $\lim_{n\to\infty} \iint_{A\cap B(0,n)} f(x,y)\,dx\,dy$ og er lik L.

Eksempel.

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\,dx$$

$$I_n = \int_{-n}^{n} e^{-\frac{x^2}{2}}\,dx$$

$$\iint_{\mathbb{R}^2} e^{-\frac{x^2+y^2}{2}}\,dx\,dy$$

$$\iint_{\mathbb{R}^2 \cap K_n} e^{-\frac{x^2+y^2}{2}}\,dx\,dy = \int_{-n}^{n}\int_{-n}^{n} e^{-\frac{x^2}{2}} \cdot e^{-\frac{y^2}{2}}\,dx\,dy$$

$$= \int_{-n}^{n} e^{-\frac{y^2}{2}} \Big(\underbrace{\int_{-n}^{n} e^{-\frac{x^2}{2}}\,dx}_{= I_n} \Big)\,dy = I_n \int_{-n}^{n} e^{-\frac{y^2}{2}}\,dy = I_n^2$$

så

$$\sqrt{\iint_{\mathbb{R}^2} e^{-\frac{x^2+y^2}{2}}\,dx\,dy} = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\,dx$$

dersom disse grensene eksisterer

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$T' = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

$$\iint_{\mathbb{R}^2 \cap B(0,n)} e^{-\frac{x^2+y^2}{2}}\,dx\,dy$$

$$\det T' = r\cos^2\theta + r\sin^2\theta = r$$

$$= \iint_{B(0,n)} e^{-\frac{r^2}{2}}\, r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{n} r e^{-\frac{r^2}{2}}\,dr\,d\theta = 2\pi \int_0^{n} r e^{-\frac{r^2}{2}}\,dr$$

$$\int_0^{2\pi} 1\cdot d\theta$$

$$u = \frac{r^2}{2}$$

$$du = r\,dr$$

$$= 2\pi \int_0^{\frac{n^2}{2}} e^{-u}\,du = 2\pi \left[-e^{-u}\right]_0^{\frac{n^2}{2}} = 2\pi\left(1 - e^{-\frac{n^2}{2}}\right)$$

$$n \to \infty, \quad e^{-\frac{n^2}{2}} \to 0$$

konvergerer

så

$$\iint_{\mathbb{R}^n} e^{-\frac{x^2+y^2}{2}}\,dx\,dy = 2\pi$$

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\,dx = \sqrt{2\pi}$$

$$\Rightarrow \quad \underline{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\,dx = 1}$$

mar 5-12:30

Skriv

$$f(x,y) = f_+(x,y) - f_-(x,y)$$

med

$$f_+(x,y) = \begin{cases} f(x,y) & f(x,y) \geq 0 \\ 0 & f(x,y) < 0 \end{cases}$$

$$f_-(x,y) = \begin{cases} -f(x,y) & f(x,y) \leq 0 \\ 0 & f(x,y) > 0 \end{cases}$$

$$\iint f(x,y)\,dx\,dy = \iint f_+(x,y)\,dx\,dy - \iint f_-(x,y)\,dx\,dy$$

dersom disse begge eksisterer!

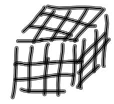
Trippel integraler

$$\iiint_R f(x,y,z)\,dx\,dy\,dz \qquad R \subseteq \mathbb{R}^3$$

Boks: $R = \{a_1 \le x \le a_2,\ b_1 \le y \le b_2,\ c_1 \le z \le c_2\}$

$$= [a_1, a_2] \times [b_1, b_2] \times [c_1, c_2]$$

$$\int_{c_1}^{c_2}\int_{b_1}^{b_2}\int_{a_1}^{a_2} f(x,y,z)\,dx\,dy\,dz$$



La Π være en partisjon av R:
det vil si en oppdeling i mindre bokser,

$$R_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$$
$$1 \le i \le n \qquad 1 \le j \le m \qquad 1 \le k \le l$$

$a_1 = x_0 < x_1 < x_2 \dots < x_n = a_2 \qquad b_1 = y_0 < y_1 < \dots < y_m = b_2$

$c_1 = z_0 < z_1 < \dots < z_l = c_2$

Anta at f er begrenset, La $m_{ijk} = \inf\{f(x,y,z) \mid (x,y,z) \in R_{ijk}\}$,

La $M_{ijk} = \sup\{f(x,y,z) \mid (x,y,z) \in R_{ijk}\}$

$$N(\Pi) = \sum_{k=1}^{l}\sum_{j=1}^{m}\sum_{i=1}^{n} m_{ijk} \cdot V(R_{ijk})$$

$V(R_{ijk}) = (a_2 - a_1)(b_2 - b_1) \cdot (c_2 - c_1)$

$$\emptyset(\Pi) = \sum\sum\sum M_{ijk}\, V(R_{ijk})$$

$$\underline{\iiint_R} f(x,y,z)\,dx\,dy\,dz = \sup_\Pi \{N(\Pi) \mid \Pi \text{ part av } R\}$$

$$\overline{\iiint_R} f(x,y,z)\,dx\,dy\,dz = \inf_\Pi \{\emptyset(\Pi) \mid \Pi \text{ part av } R\}$$

f er integrerbar over $R \subseteq \mathbb{R}^3$ dersom

$$\underline{\iiint} = \overline{\iiint}, \text{ og i så fall}$$

skrives vi $\displaystyle\iiint_R f(x,y,z)\,dx\,dy\,dz = \underline{\iiint} = \overline{\iiint}$.

Hvis f er integrerbar på R så er

$$\iiint_R f(x,y,z)\,dx\,dy\,dz = \int_{c_1}^{c_2}\int_{b_1}^{b_2}\int_{a_1}^{a_2} f(x,y,z)\,dx\,dy\,dz$$

og rekkefølgen kan byttes om.

Eks

$$\int_2^3\int_1^2\int_0^1 xyz \; dx\, dy\, dz = \int_2^3\int_1^2 yz \underbrace{\left[\tfrac{1}{2}x^2\right]_0^1}_{=\frac{1}{2}} dy\, dz$$

$$R = [0,1]\times[1,2]\times[2,3] \subseteq \mathbb{R}^3$$

$$= \frac{1}{2}\int_2^3 z \underbrace{\left[\tfrac{1}{2}y^2\right]_1^2}_{=2-\frac{1}{2}} dz$$

$$= \frac{1}{2}\cdot\frac{3}{2}\left[\tfrac{1}{2}z^2\right]_2^3 = \frac{3}{4}\cdot\frac{1}{2}(9-4)$$

$$= \frac{15}{8}.$$

$$= \int_0^1\int_1^2\int_2^3 xyz\; dz\, dy\, dx$$

$$= \frac{5}{2}\int_0^1\int_1^2 xy\; dy\, dx = \frac{5}{2}\cdot\frac{3}{2}\int_0^1 x\, dx$$

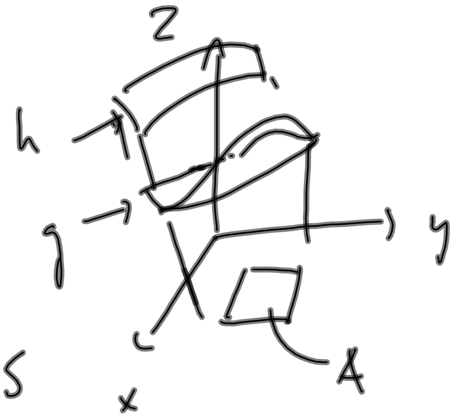
$$= \underline{\frac{15}{8}}$$

$A \subseteq \mathbb{R}^2$ A lukket og begrenset $(\subseteq [a_1,a_2] \times [b_1,b_2])$

$h, g : A \to \mathbb{R}$ kontinuerlige funksjoner

anta at $g(x,y) \le h(x,y) \quad \forall\, x,y \in A.$

$$S = \{ (x,y,z) \mid (x,y) \in A \text{ og } g(x,y) \le z \le h(x,y) \}$$



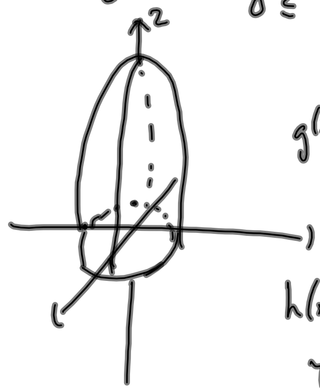
Hvis f er kontinuerlig på S så er f integrerbar på S og

$$\iiint_S f(x,y,z)\, dx\, dy\, dz = \iint_A \int_{g(x,y)}^{h(x,y)} f(x,y,z)\, dz\, dx\, dy$$

Eks:

$$\iiint_S x \, dx\,dy\,dz$$

$S = \{(x,y,z) \mid x^2+y^2 \le 4,\ 0 \le z \le 4-x^2-y^2\}$



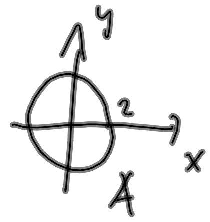
$g(x,y) = 0$

$h(x,y) = 4-x^2-y^2$

$$= \iint_A \int_0^{4-x^2-y^2} x \, dz \, dx\,dy$$

$x^2+y^2 \le 4$

$$= \iint_A x \Big[z \Big]_0^{4-x^2-y^2} dx\,dy$$



$$= \iint_A 4x - x^3 - xy^2 \, dx\,dy$$

$x = r\cos\theta$

$y = r\sin\theta$

$$= \int_0^{2\pi}\int_0^2 \left(4 r\cos\theta - r^3\cos^3\theta - r^3\cos\theta\sin^2\theta\right) \cdot r \, dr\, d\theta$$

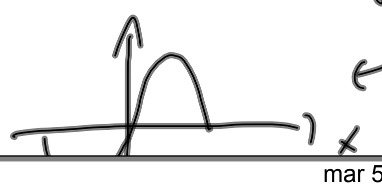
$\cos\theta - \cos\theta\sin^2\theta$

$$= \int_0^{2\pi} \left(\frac{4}{3}\cdot 2^3\right)\cdot\cos\theta - \left(\frac{1}{5}\cdot 2^5\right)\cos^3\theta - \left(\frac{1}{5}\cdot 2^5\right)\cos\theta\sin^2\theta \, d\theta$$

$$= \left[\left(\frac{4}{3}\cdot 2^3\right)(-\sin\theta)\right]_0^{2\pi} - \left[\left(\frac{1}{5}2^5\right)\left((-\sin\theta) - \frac{1}{3}\sin^3\theta\right)\right]_0^{2\pi}$$

$$- \left[\left(\frac{1}{5}2^5\right)\left(\frac{1}{3}\sin^3\theta\right)\right]_0^{2\pi} = \underline{0}$$

Siden S er symmetrisk om y,z-planet og $x \to -x$ ved speiling om dette planet er integralet $= 0$.



← tenk på denne.