Trippel integral

Skifte av variabler:

$$\mathbb{R}^3 \supseteq U \xrightarrow{T} \mathbb{R}^3$$

injektiv, kontinuerlige partiell deriverte

$$U \longrightarrow T(U)$$

$$\det(T') \neq 0 \text{ på hele } U$$

$$(u,v,w) \longmapsto (x,y,z)$$

$x(u,v,w)$

$y(u,v,w)$

$z(u,v,w)$

$$T' = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial w} & \frac{\partial z}{\partial w} \end{pmatrix}$$

Dersom $D \subseteq U$ er Jordan målbar og f kontinuerlig, så er

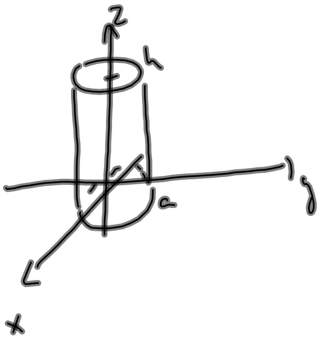
$$\iiint_{T(D)} f(x,y,z)\,dx\,dy\,dz = \iiint_{D} f(T(u,v,v))\,|\det(T')|\,du\,dv\,dw$$

Analogt med: $x = x(u) : [a,b] \longrightarrow [x(a), x(b)]$

$$\int_{x(a)}^{x(b)} f(x)\,dx = \int_a^b f(x(u))\,x'(u)\,du$$

$$\frac{dx}{du} = x'(u)$$

Eks Sylinder koordinater.



$$S = \{(x,y,z) \mid 0 \leq x^2+y^2 \leq a^2,\ 0 \leq z \leq h\}$$

$$S' = \{(r,\theta,z) \mid 0 \leq r \leq a,\ 0 \leq \theta \leq 2\pi,\ 0 \leq z \leq h\}$$

$$= [0,a] \times [0,2\pi] \times [0,h]$$

$$T = \begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

$$S' \xrightarrow{\qquad\qquad} S$$

$$\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \vdots & & \vdots \\ \vdots & & \vdots \end{pmatrix}$$

$$T' = \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \underline{\det T'} = r(\cos^2\theta + \sin^2\theta)$$

$$= \underline{r}$$

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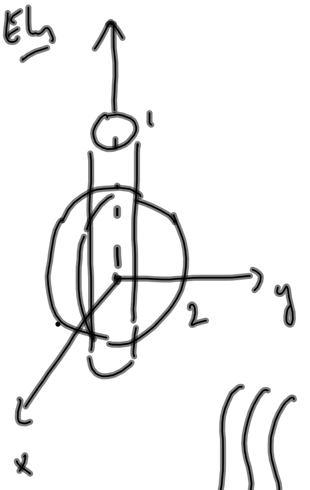
Ex



$$S = \{(x,y,z) \mid x^2+y^2 \le 1,\ x^2+y^2+z^2 \le 4,\ z \ge 0\}$$

$$S' = \{(r,\theta,z) \mid 0 \le r \le 1,\ 0 \le \theta \le 2\pi,\ 0 \le z \le \sqrt{4-r^2}\}$$

$$x = r\cos\theta \quad y = r\sin\theta \quad z = z$$

$$\iiint_S x^2 z \, dx\,dy\,dz = \iiint_{S'} r^2\cos^2\theta \cdot z \; r\,dr\,d\theta\,dz$$

$$= \int_0^{2\pi}\int_0^1\int_0^{\sqrt{4-r^2}} r^3 \cdot z \cdot \cos^2\theta \; dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 r^3\cos^2\theta \left[\tfrac{1}{2}z^2\right]_0^{\sqrt{4-r^2}} dr\,d\theta$$

$$= \int_0^{2\pi}\cos^2\theta \int_0^1 \tfrac{1}{2}r^3(4-r^2)\,dr\,d\theta$$

$$= \int_0^{2\pi}\cos^2\theta \left[\tfrac{2}{4}r^4 - \tfrac{1}{12}r^6\right]_0^1 d\theta$$

$$= \frac{5}{12}\int_0^{2\pi} \tfrac{1}{2}\cos 2\theta + \tfrac{1}{2}\; d\theta = \underline{\frac{5}{12}\pi}$$

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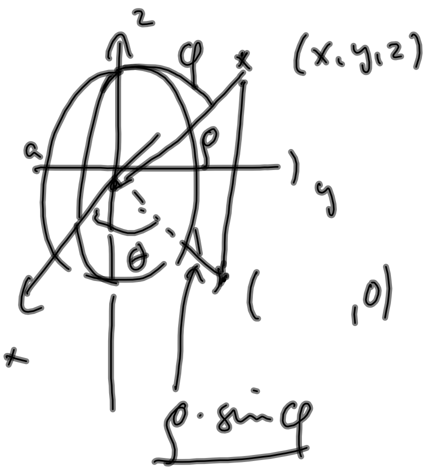
Kulekoordinater:



$$S = \left\{ (x,y,z) \;\middle|\; x^2 + y^2 + z^2 \leq a^2 \right\}$$

$$S' = \left\{ (\rho, \varphi, \theta) \;\middle|\; 0 \leq \rho \leq a,\ 0 \leq \varphi \leq \pi,\ 0 \leq \theta \leq 2\pi \right\}$$

$$T: \begin{cases} x = \rho \sin\varphi \cdot \cos\theta \\ y = \rho \sin\varphi \sin\theta \\ z = \rho \cdot \cos\varphi \end{cases}$$

$$T' = \begin{pmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \varphi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \varphi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \varphi} & \frac{\partial z}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \sin\varphi\cos\theta & \rho\cos\varphi\cos\theta & -\rho\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \rho\cos\varphi\sin\theta & \rho\sin\varphi\cos\theta \\ \cos\varphi & -\rho\sin\varphi & 0 \end{pmatrix}$$

$$\det T' = \rho^2 \sin\varphi \begin{vmatrix} \sin\varphi\cos\theta & \cos\varphi\cos\theta & -\sin\theta \\ \sin\varphi\sin\theta & \cos\varphi\sin\theta & \cos\theta \\ \cos\varphi & -\sin\varphi & 0 \end{vmatrix}$$

$$\begin{vmatrix} ab & d \\ ac & e \end{vmatrix} = a \begin{vmatrix} b & d \\ c & e \end{vmatrix}$$

$$= \rho^2 \sin\varphi \Big(\cos\varphi \big(\underbrace{\cos\varphi \cos^2\theta + \cos\varphi \sin^2\theta}_{=\cos\varphi} \big) + \sin\varphi \big(\underbrace{\sin\varphi \cos^2\theta + \sin\varphi \sin^2\theta}_{=\sin\varphi} \big) \Big)$$

$$= \rho^2 \sin\varphi \left(\cos^2\varphi + \sin^2\varphi \right)$$

$$= \underline{\rho^2 \sin\varphi} \quad \geq 0 \qquad 0 \leq \rho \leq \pi$$

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Eksempler

$$\iiint_S dV$$

Volumet av S : $\mathrm{Vol}(S) = \iiint_S 1 \cdot dV$

Kule (radius 1) : $\underline{K}$

$$K := \left\{ (\rho, \varphi, \theta) \;\middle|\; \begin{array}{l} \rho \le 1 \\ 0 \le \varphi \le \pi \\ 0 \le \theta \le 2\pi \end{array} \right\}$$

$$\underline{dV} = dx\,dy\,dz = r\,dr\,d\theta\,dz = \rho^2 \sin\varphi\,d\rho\,d\varphi\,d\theta$$

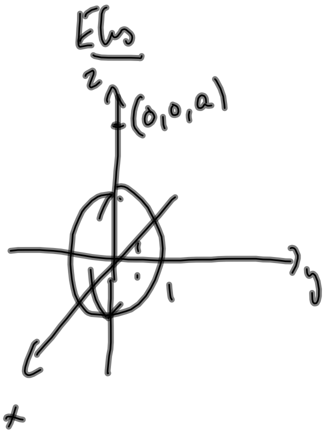
$$\mathrm{Vol}(K) = \iiint_K dV$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{1} \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta$$

$$= 2\pi \int_0^{\pi}\int_0^{1} \rho^2 \sin\varphi \, d\rho\, d\varphi = 2\pi \int_0^{\pi} \sin\varphi \left[\tfrac{1}{3}\rho^3\right]_0^1 d\varphi$$

$$= \frac{2\pi}{3}\int_0^{\pi} \sin\varphi\, d\varphi = \frac{2\pi}{3}\, 2 = \underline{\frac{4\pi}{3}}$$

Eks



I_a = gjennomsnittlig avstand fra $(0,0,a)$ til pkt i kula.

$$I_a = \frac{1}{V(K)}\iiint_K \sqrt{(x-0)^2+(y-0)^2+(z-a)^2}\;\underline{dx\,dy\,dz}$$

$$" \frac{1}{V(K)}\iiint_K f\,dV$$

$$I_a = \frac{3}{4\pi}\iiint_K \sqrt{x^2+y^2+z^2-2az+a^2}\,dx\,dy\,dz$$

$$= \frac{3}{4\pi}\int_0^{2\pi}\int_0^{\pi}\int_0^1 \sqrt{\rho^2-2a\rho\cos\varphi+a^2}\;\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

$$= \frac{3}{2}\int_0^{\pi}\int_0^1 \sqrt{\rho^2-2a\rho\cos\varphi+a^2}\;\rho^2\sin\varphi\,d\rho\,d\varphi$$

$$u(\varphi) = \rho^2-2a\rho\cos\varphi+a^2 \qquad \frac{du}{d\varphi} = 2a\rho\sin\varphi$$

$$du = 2a\rho\sin\varphi\,d\varphi$$

$$= \frac{3}{2}\int_0^1 \rho \int_{\rho^2-2a\rho+a^2}^{\rho^2+2a\rho+a^2} \sqrt{u}\;\frac{1}{2a}\,du\,d\rho$$

$$\underline{\rho<a}\,!$$

$$= \frac{3}{2}\int_0^1 \frac{\rho}{2a}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_{(\rho-a)^2}^{(\rho+a)^2} d\rho \qquad \left((\rho-a)^2\right)^{\frac{3}{2}} = (a-\rho)$$

$$= \frac{3}{2}\cdot\frac{2}{3}\int_0^1 \frac{\rho}{2a}\left((\rho+a)^3-(a-\rho)^3\right)d\rho$$

$$= \frac{1}{2a}\int_0^1 \rho\left(2\rho^3+2\cdot 3\rho a^2\right)d\rho = \frac{1}{2a}\left[\frac{2}{5}\rho^5+2a^2\rho^3\right]_0^1$$

$$= \frac{1}{2a}\left(\frac{2}{5}+2a^2\right) = \frac{a}{\;}+a$$

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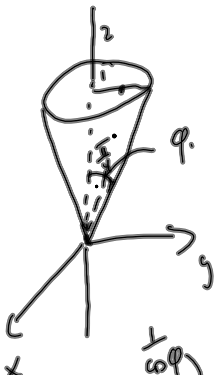
Eks

$$V(S) = \iiint_S 1 \, dV \qquad S = \{(\rho, \varphi, \theta) \mid 0 \le \theta \le 2\pi,\ 0 \le \varphi \le \tfrac{\pi}{4},\ 0 \le \rho \le \tfrac{1}{\cos\varphi}\}$$



Masse $= M(S)$

$M =$ tetthet $\cdot V$

$dM = t(x,y,z) \cdot dV$

$$M(S) = \iiint_S dM = \iiint_S t \, dV$$

$$t = k \cdot \rho$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\frac{1}{\cos\varphi}} k \cdot \rho \cdot \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$= 2\pi \int_0^{\frac{\pi}{4}} \left[\tfrac{1}{4} k \rho^4 \sin\varphi\right]_0^{\frac{1}{\cos\varphi}} d\varphi$$

$$= 2\pi \tfrac{1}{4} k \int_0^{\frac{\pi}{4}} \frac{1}{\cos^4\varphi} \cdot \sin\varphi \, d\varphi = \frac{\pi}{2} \cdot k \left[-(-3)\frac{1}{\cos^3\varphi}\right]_0^{\frac{\pi}{4}}$$

$$= \frac{3\pi k}{2}\left(2\sqrt{2} - 1\right)$$

Tyngdepkt: $(\bar{x}, \bar{y}, \bar{z})$

$$\bar{x} = \frac{1}{M} \iiint_S x \cdot t \cdot dV = \frac{1}{M} \iiint_S x \, dM$$

$\bar{y} =$ $\qquad \bar{z}$

I eks: $\bar{x} = 0 \quad \bar{y} = 0$ (av symmetri).

$$\bar{z} = \frac{1}{M} \iiint z \cdot t \, dV = \iiint k \rho^3 \sin\varphi \cdot \rho \cos\varphi \, d\theta \, d\rho \, d\varphi$$

$$= \frac{2\pi k}{M} \int_0^{\frac{\pi}{4}} \frac{1}{5} \cdot \frac{1}{\cos^5\varphi} \cdot \sin\varphi \cos\varphi \, d\varphi$$

$$= \frac{2\pi}{15}\left(2\sqrt{2} - 1\right) \cdot \frac{1}{M}$$

$$\Rightarrow \bar{z} = \frac{4}{5}$$

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